

# CROSSED PRODUCTS OF $k$ -GRAPH $C^*$ -ALGEBRAS BY $\mathbb{Z}^l$ .

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**ABSTRACT.** An action of  $\mathbb{Z}^l$  by automorphisms of a  $k$ -graph induces an action of  $\mathbb{Z}^l$  by automorphisms of the corresponding  $k$ -graph  $C^*$ -algebra. We show how to construct a  $(k+l)$ -graph whose  $C^*$ -algebra coincides with the crossed product of the original  $k$ -graph algebra by  $\mathbb{Z}^l$ . We then investigate the structure of the crossed-product  $C^*$ -algebra.

## 1. INTRODUCTION

In recent years, much attention has been paid to graph algebras and their higher-rank analogues as models for classifiable  $C^*$ -algebras (see [17] for an overview of the subject). What makes these models so attractive is the ability to trade information back and forth between the underlying combinatorial object and the associated  $C^*$ -algebra. This program is quite advanced for graph algebras (see, for example, [7, 8, 11, 22]). However, higher-rank graphs are a more recent development and have more complicated combinatorial properties than ordinary graphs. Consequently, many important structural questions regarding higher-rank graph algebras remain unanswered. In particular, while the general theory of  $k$ -graph  $C^*$ -algebras is quickly catching up with that of graph  $C^*$ -algebras (see for example [4, 5, 10, 19, 21]), there remains a dearth of tractable examples in the higher-rank setting.

A construction which *has* been successfully generalised from the setting of graphs to that of higher-rank graphs is the skew-product construction (see [10, 13]). This construction allows us to realise certain crossed products of  $k$ -graph algebras by coactions of groups  $G$  as  $k$ -graph algebras in their own right. Specifically, suppose that the coaction  $\delta$  arises from a functor  $c$  from the  $k$ -graph  $\Lambda$  to the group  $G$  (that is, a function which takes composition in  $\Lambda$  to multiplication in  $G$ ). Then we may form a skew-product  $k$ -graph  $\Lambda \times_c G$ , and the coaction crossed product  $C^*(\Lambda) \times_\delta G$  is canonically isomorphic to  $C^*(\Lambda \times_c G)$ .

Results of [9, 10] show that we can also realise certain crossed products of  $k$ -graph  $C^*$ -algebras by *actions* of groups as  $k$ -graph  $C^*$ -algebras. This is achieved using the quotient-graph construction. Specifically, an action  $\alpha$  of a group  $G$  by automorphisms of a  $k$ -graph  $\Lambda$  induces an action  $\tilde{\alpha}$  of  $G$  by automorphisms of  $C^*(\Lambda)$ . If  $\alpha$  is free in the sense that no nontrivial group element fixes any vertex, then one

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may form the quotient  $k$ -graph  $\Lambda/G$  whose morphisms are the orbits of morphisms of  $\Lambda$  under the action of  $G$ . In this situation, the  $C^*$ -algebras  $C^*(\Lambda) \rtimes_{\tilde{\alpha}} G$  and  $C^*(\Lambda/G)$  are Morita equivalent [10, Theorem 5.7]. If the action  $\alpha$  is not free, however, [13, page 176] shows that  $\Lambda/G$  may not be a  $k$ -graph because composition may not be well-defined, so the approach of [9, 10] is not applicable. Moreover, notice that whereas there is an isomorphism between a skew-product  $k$ -graph  $C^*$ -algebra and the associated coaction crossed product  $C^*$ -algebra, the corresponding result for actions using the quotient  $k$ -graph construction yields only a Morita equivalence.

In this article we describe a class of  $(k+l)$ -graphs whose  $C^*$ -algebras are isomorphic to crossed products of  $k$ -graph algebras by  $\mathbb{Z}^l$ . Given an action  $\alpha$  of  $\mathbb{Z}^l$  on a  $k$ -graph  $\Lambda$ , we construct a  $(k+l)$ -graph  $\Lambda \times_{\alpha} \mathbb{Z}^l$  which, as the notation suggests, can profitably be thought of as the crossed product of  $\Lambda$  by  $\alpha$ . We show that  $\alpha$  induces an action  $\tilde{\alpha}$  of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$  and that the higher-rank graph  $C^*$ -algebra  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  coincides with the crossed-product  $C^*$ -algebra  $C^*(\Lambda) \rtimes_{\tilde{\alpha}} \mathbb{Z}^l$ .

It is noteworthy that our results do not require that  $\alpha$  should be free, and we obtain an isomorphism rather than a Morita equivalence. Moreover, there is good evidence to suggest that our notion of a crossed-product  $k$ -graph is a reasonable one. For example, we combine our construction with the skew-product construction of [10, 13] to obtain a realisation of Takai duality at the level of higher-rank graphs (see Section 3.2).

The identification of  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  with  $C^*(\Lambda) \rtimes_{\tilde{\alpha}} \mathbb{Z}^l$  allows us to study the crossed-product  $C^*$ -algebra using the theory of graph  $C^*$ -algebras, and in particular to formulate necessary and sufficient conditions for simplicity of the crossed product. It also allows us to study the  $(k+l)$ -graph  $C^*$ -algebra using the theory of crossed-product  $C^*$ -algebras; for example, when  $l = 1$ , our construction lends itself to analysis of the  $K$ -theory of  $C^*(\Lambda \times_{\alpha} \mathbb{Z})$  via the Pimsner-Voiculescu exact sequence.

The paper is organised as follows. In Section 2, we introduce the notation and conventions we will use throughout. We open Section 3 by describing our construction of a  $(k+l)$ -graph  $\Lambda \times_{\alpha} \mathbb{Z}^l$  from an action  $\alpha$  of  $\mathbb{Z}^l$  on a  $k$ -graph  $\Lambda$ . We establish that the action  $\alpha$  of  $\mathbb{Z}^l$  on  $\Lambda$  induces an action  $\tilde{\alpha}$  of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$ , and then prove our first main result, Theorem 3.5: the  $(k+l)$ -graph algebra  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  is canonically isomorphic to the crossed product  $C^*$ -algebra  $C^*(\Lambda) \rtimes_{\tilde{\alpha}} \mathbb{Z}^l$ .

In Section 4, we use the recent results of [21] which characterise simplicity of higher-rank graph algebras to decide when  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  is simple in terms of properties of  $\Lambda$  and the action  $\alpha$ . In Section 5, we recast the results of Section 4 in terms of features of the  $C^*$ -algebra  $C^*(\Lambda)$  and of properties of the induced action  $\tilde{\alpha}$  of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$ .

We conclude in section 6 with an application of our results to the calculation of  $K$ -theory for certain examples. Specifically, we consider a 1-graph  $E$  endowed with an action  $\alpha$  of  $\mathbb{Z}$  so that  $\tilde{\alpha}$  is an action of  $\mathbb{Z}$  on  $C^*(E)$ . If either of the  $K$ -groups of  $C^*(E)$  is trivial, we may use the Pimsner-Voiculescu exact sequence in  $K$ -theory to calculate the  $K$ -groups of  $C^*(E \times_{\alpha} \mathbb{Z})$ .

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## 2. PRELIMINARIES

We first recall the notation and conventions used for  $k$ -graphs. For more details see [17, 19].

**2.1. The semigroup  $\mathbb{N}^k$ .** We write  $\mathbb{N}$  for the semigroup  $\{0, 1, 2, \dots\}$  under addition. We regard  $\mathbb{N}^k$  as a semigroup under addition with identity element denoted  $0_k$ . When convenient, we regard the semigroup  $\mathbb{N}^k$  as (the morphisms of) a category with a single object (the addition operator on  $\mathbb{N}^k$  is viewed as a composition map).

We denote the canonical generators of  $\mathbb{N}^k$  by  $e_1, \dots, e_k$ , and for  $n \in \mathbb{N}^k$  and  $1 \leq i \leq k$ , we write  $n_i$  for the  $i^{\text{th}}$  coordinate of  $n$ , so that  $n = \sum_{i=1}^k n_i e_i$ . Fix  $m, n \in \mathbb{N}^k$ . We write  $m \leq n$  if  $m_i \leq n_i$  for all  $i$ . We denote by  $m \vee n$  the coordinatewise maximum of  $m$  and  $n$ , and  $m \wedge n$  the coordinatewise minimum. In particular, we have  $m \wedge n \leq m, n \leq m \vee n$ .

We shall often and without comment identify  $\mathbb{N}^{k+l}$  with  $\mathbb{N}^k \times \mathbb{N}^l$ . In particular, we write  $(p, m) \in \mathbb{N}^{k+l}$  to indicate that  $(p, m)$  is the element of  $\mathbb{N}^{k+l}$  whose first  $k$  coordinates are those of  $p \in \mathbb{N}^k$  and whose last  $l$  coordinates are those of  $m \in \mathbb{N}^l$ .

**2.2. Higher-rank graphs.** Recall from [10, Definition 1.1] that a  $k$ -graph is a countable category  $\Lambda$  together with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$  which satisfies the factorisation property: if  $\lambda \in \Lambda$  with  $d(\lambda) = m + n$ , then there exist unique elements  $\mu \in d^{-1}(m)$  and  $\nu \in d^{-1}(n)$  such that  $\lambda = \mu\nu$ . We call  $d$  the *degree functor* and regard it as a higher-rank analogue of length. An argument involving the factorisation property shows that  $v \mapsto \text{id}_v$  is a bijection between the objects of  $\Lambda$  and the morphisms of degree  $0_k$ . We use this to identify the two, and we regard  $\Lambda$  as a collection of morphisms only.

For  $n \in \mathbb{N}^k$ , we denote  $d^{-1}(n)$  by  $\Lambda^n$ . We call the elements of  $\Lambda$  *paths* and the elements of  $\Lambda^{0_k}$  *vertices*. If  $\lambda \in \Lambda^p$  and  $0_k \leq m \leq n \leq p$  then we denote by  $\lambda(0_k, m)$ ,  $\lambda(m, n)$  and  $\lambda(n, p)$  the unique elements of  $\Lambda^m$ ,  $\Lambda^{n-m}$  and  $\Lambda^{p-n}$  satisfying  $\lambda = \lambda(0_k, m)\lambda(m, n)\lambda(n, p)$  (the existence and uniqueness of these morphisms follows from two applications of the factorisation property.)

For  $\mu, \nu \in \Lambda$  we call  $\lambda$  a *common extension* of  $\mu$  and  $\nu$  if  $\lambda = \mu\alpha = \nu\beta$  for some  $\alpha, \beta \in \Lambda$ . If  $\lambda$  is a common extension of  $\mu$  and  $\nu$  then we must have  $d(\lambda) \geq d(\mu) \vee d(\nu)$ . We call  $\lambda$  a *minimal common extension* of  $\mu$  and  $\nu$  if it is a common extension satisfying  $d(\lambda) = d(\mu) \vee d(\nu)$ . We write  $\text{MCE}(\mu, \nu)$  for the set of all minimal common extensions of  $\mu$  and  $\nu$ . We say that  $\Lambda$  is *finitely aligned* if  $\text{MCE}(\mu, \nu)$  is finite (possibly empty) for all  $\mu, \nu \in \Lambda$ . We write  $\Lambda^{\min}(\mu, \nu)$  for the set  $\{(\alpha, \beta) : \mu\alpha = \nu\beta \in \text{MCE}(\mu, \nu)\}$ .

If  $S$  is a subset of  $\Lambda$  and  $v \in \Lambda^{0_k}$ , we write  $vS$  for the set  $S \cap r^{-1}(v)$  and we write  $Sv$  for  $S \cap s^{-1}(v)$ . We say that  $\Lambda$  is *row-finite* if  $|v\Lambda^n| < \infty$  for all  $v \in \Lambda^{0_k}$  and  $n \in \mathbb{N}^k$ , and we say that  $\Lambda$  has *no sources* if  $v\Lambda^n \neq \emptyset$  for all  $v \in \Lambda^{0_k}$  and  $n \in \mathbb{N}^k$ .

Given a vertex  $v \in \Lambda^{0_k}$  and a subset  $F$  of  $v\Lambda$ , we say  $F$  is *exhaustive* if for every  $\mu \in v\Lambda$  there exists  $\nu \in F$  such that  $\text{MCE}(\mu, \nu) \neq \emptyset$ . We say an exhaustive set  $F \subset v\Lambda$  is *finite exhaustive* if  $|F| < \infty$ . If  $\Lambda$  has no sources, then  $v\Lambda^n$  is exhaustive for all  $v \in \Lambda^{0_k}$  and  $n \in \mathbb{N}^k$ , so if  $\Lambda$  is row-finite and has no sources, then  $v\Lambda^n$  is always finite exhaustive.

**2.3. The universal  $C^*$ -algebra  $C^*(\Lambda)$ .** As in [19], given a finitely aligned  $k$ -graph  $\Lambda$ , a Cuntz-Krieger  $\Lambda$ -family is a set  $\{t_\lambda : \lambda \in \Lambda\}$  of partial isometries satisfying

(TCK1)  $\{t_v : v \in \Lambda^{0_k}\}$  is a set of mutually orthogonal projections.

(TCK2)  $t_\mu t_\nu = t_{\mu\nu}$  whenever  $r(\nu) = s(\mu)$ .

(TCK3)  $t_\mu^* t_\nu = \sum_{(\xi, \eta) \in \Lambda^{\min}(\mu, \nu)} t_\xi t_\eta^*$ .

(CK)  $\prod_{\lambda \in F} (t_v - t_\lambda t_\lambda^*) = 0$  for each  $v \in \Lambda^{0_k}$  and each finite exhaustive  $F \subset v\Lambda$ .

By  $t_\Lambda$ , we mean the Cuntz-Krieger  $\Lambda$ -family  $\{t_\lambda : \lambda \in \Lambda\}$  as a whole.

General results of Blackadar (see [2]) imply that there is a  $C^*$ -algebra  $C^*(\Lambda)$  (unique up to canonical isomorphism) generated by a Cuntz-Krieger  $\Lambda$ -family  $s_\Lambda$  which is universal in the sense that for any other Cuntz-Krieger  $\Lambda$ -family  $t_\Lambda$  there is a homomorphism  $\pi_t : C^*(\Lambda) \rightarrow C^*(t_\Lambda)$  satisfying  $\pi_t(s_\lambda) = t_\lambda$  for all  $\lambda \in \Lambda$ .

For  $z \in \mathbb{T}^k$  and  $n \in \mathbb{Z}^k$ , we employ multi-index notation and write  $z^n$  for the product  $\prod_{i=1}^k z_i^{n_i} \in \mathbb{T}$ . Using the universal property of  $C^*(\Lambda)$  one can check that there is a strongly continuous action  $\gamma$  of  $\mathbb{T}^k$  on  $C^*(\Lambda)$  satisfying  $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$  for all  $\lambda \in \Lambda$  and  $z \in \mathbb{T}^k$ . This action  $\gamma$  is called the *gauge action*.

**2.4. Graph morphisms and infinite paths.** Given  $k$ -graphs  $\Lambda$  and  $\Gamma$ , a  $k$ -graph *morphism*  $\phi : \Lambda \rightarrow \Gamma$  is a functor from  $\Lambda$  to  $\Gamma$  which respects the degree maps. A bijective  $k$ -graph morphism is simply called an isomorphism. An isomorphism  $\phi : \Lambda \rightarrow \Lambda$  is called an automorphism of  $\Lambda$ .

To discuss infinite paths in  $k$ -graphs, we must first introduce the  $k$ -graph  $\Omega_k$ . For  $k \geq 1$ ,  $\Omega_k$  is the  $k$ -graph  $\{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$  with structure maps  $r(m, n) := (m, m)$ ,  $s(m, n) := (n, n)$ ,  $(m, n)(n, p) := (m, p)$  and  $d(m, n) := n - m$ . We typically denote the element  $(m, m)$  of  $\Omega_k^{0_k}$  by  $m$ .

An *infinite path* in a  $k$ -graph  $\Lambda$  is a graph morphism  $x : \Omega_k \rightarrow \Lambda$ . We denote the collection of all infinite paths in  $\Lambda$  by  $\Lambda^\infty$ . For any cofinal sequence  $(m_i)_{i=1}^\infty \subset \mathbb{N}^k$  such that  $m_0 = 0_k$ , a pair of infinite paths  $x, y$  are equal if and only if  $x(m_i, m_{i+1}) = y(m_i, m_{i+1})$  for all  $i$ . Hence, given a cofinal sequence  $(m_i)_{i=1}^\infty \subset \mathbb{N}^k$ , we may view an infinite path  $x$  as the infinite composition of finite paths  $x = x(0_k, m_1)x(m_1, m_2)x(m_2, m_3) \cdots$ . In keeping with this, we regard  $x(0_k)$  as the range of  $x$  and denote it  $r(x)$ .

Fix  $x \in \Lambda^\infty$ . For each  $\lambda \in \Lambda r(x)$  there is a unique infinite path  $\lambda x$  satisfying  $(\lambda x)(0_k, d(\lambda)) = \lambda$  and  $(\lambda x)(d(\lambda), d(\lambda) + m) = x(0_k, m)$  for all  $m \in \mathbb{N}^k$ . We write  $\lambda \Lambda^\infty$  for the set  $\{\lambda x : x \in \Lambda^\infty, r(x) = s(\lambda)\} \subset \Lambda^\infty$ , and call this the *cylinder set associated to  $\lambda$* . For each  $m \in \mathbb{N}^k$ , there is a unique infinite path  $\sigma^m(x)$  such that  $\sigma^m(x)(0_k, n) = x(m, m+n)$  for all  $n \in \mathbb{N}^k$ . Note that  $\sigma^{d(\lambda)}(\lambda x) = x = x(0, m)\sigma^m(x)$  for each  $\lambda \in \Lambda r(x)$  and each  $m \in \mathbb{N}^k$ .

**2.5. The  $k$ -graph  $\Delta_k$ .** Related to infinite paths and the  $k$ -graph  $\Omega_k$  is the two-sided version  $\Delta_k$  of  $\Omega_k$ . For an integer  $k \geq 1$ , we write  $\Delta_k$  for the  $k$ -graph  $\Delta_k := \{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k : m \leq n\}$  with  $r(m, n) = m$ ,  $s(m, n) = n$ ,  $d(m, n) = n - m$  and  $(m, n)(n, p) := (m, p)$ .

Note that  $\Omega_k$  is isomorphic to the sub- $k$ -graph of  $\Delta_k$  consisting of elements  $(m, n)$  such that  $m \geq 0$ . As with  $\Omega_k$  we usually denote a vertex  $(m, m)$  of  $\Delta_k$  by  $m$ .

**2.6. Skeletons.** One can completely describe a  $k$ -graph using its skeleton, which consists of a  $k$ -coloured graph  $E$  and a list of factorisation rules  $F$ . We outline this construction here, but see [17, pp. 90–91] or [18, Section 2] for more detail.

By a  $k$ -coloured graph, we mean a 5-tuple  $(E^0, E^1, r, s, \text{col})$  where  $(E^0, E^1, r, s)$  is a directed graph, and  $\text{col} : E^1 \rightarrow \{1, \dots, k\}$  is the *colour* function. For convenience, we denote  $\text{col}^{-1}(i) \subset E^1$  by  $E_i^1$ .

To each  $k$ -graph  $\Lambda$  we associate a  $k$ -coloured graph  $E_\Lambda = (\Lambda^0, \bigsqcup_{i=1}^k \Lambda^{e_i}, r, s, \text{col})$  where the range and source maps  $r, s$  are inherited from  $\Lambda$ , and where  $\text{col}(f) = i$  if and only if  $f \in \Lambda^{e_i}$ . In the examples in this article, two colours will suffice: edges of degree  $e_1$  will be thought of as blue (and drawn using solid lines), and edges of degree  $e_2$  will be thought of as red (and drawn using dashed lines).

Let  $1 \leq i < j \leq k$ , and suppose  $f, g \in E_\Lambda^1$  with  $\text{col}(f) = i$ ,  $\text{col}(g) = j$  and  $s(f) = r(g)$ . Then  $fg \in \Lambda^{e_i + e_j}$ , so the factorisation property in  $\Lambda$  applied with  $d(fg) = e_j + e_i$  ensures that there are unique edges  $g', f' \in E_\Lambda^1$  such that  $\text{col}(f') = i$ ,  $\text{col}(g') = j$ ,  $s(g') = r(f')$ , and  $fg = g'f'$  in  $\Lambda$ . The list of factorisation rules  $F_\Lambda$  associated to  $\Lambda$  is the complete set of equalities  $fg = g'f'$  obtained this way. The skeleton of  $\Lambda$  is the pair  $(E_\Lambda, F_\Lambda)$ .

Conversely, let  $E = (E^0, E^1, r, s, \text{col})$  be a  $k$ -coloured directed graph. Let  $F$  be a list of equalities of the form  $fg = g'f'$  where  $f, f' \in E_i^1$ ,  $g, g' \in E_j^1$ ,  $i < j$ ,  $s(f) = r(g)$  and  $s(g') = r(f')$ . We say that  $F$  is *permissible* if it satisfies two conditions. The first condition is essentially the factorisation property for bi-coloured paths of length 2:

- (1) the factorisation rules determine bijections  $E_i^1 \times_{E^0} E_j^1 \rightarrow E_j^1 \times_{E^0} E_i^1$  for  $i < j$ , where  $E_i^1 \times_{E^0} E_j^1$  is the fibred product  $\{(f, g) \in E_i^1 \times E_j^1 : s(f) = r(g)\}$ . That is, each bi-coloured path in  $E$  appears in exactly one factorisation rule.

To state the second rule, observe that if  $(f, g, h) \in E_i^1 \times_{E^0} E_j^1 \times_{E^0} E_l^1$  (where  $i, j, l \in \{1, \dots, k\}$  are distinct), then (1) gives unique edges  $f^1, f^2 \in E_i^1$ ,  $g^1, g^2 \in E_j^1$  and  $h^1, h^2 \in E_l^1$  such that

$$fgh = fh^1g^1 = h^2f^1g^1 = h^2f^2g^2.$$

Likewise there are unique edges  $f_1, f_2 \in E_i^1$ ,  $g_1, g_2 \in E_j^1$  and  $h_1, h_2 \in E_l^1$  such that

$$fgh = g_1f_1h = g_1h_1f_2 = h_2g_2f_2.$$

The collection  $F$  of factorisation rules is permissible if it satisfies (1) and

- (2) The factorisation rules are associative: for every  $(f, g, h) \in E_i^1 \times_{E^0} E_j^1 \times_{E^0} E_l^1$  such that  $i, j, l \in \{1, \dots, k\}$  are distinct, the two different ways of reversing

the colours in the path  $fgh$  discussed above agree:  $f^2 = f_2$ ,  $g^2 = g_2$  and  $h^2 = h_2$ .

The pair  $(E, F)$  is then called a skeleton. The results of [6] imply that the map  $\Lambda \mapsto (E_\Lambda, F_\Lambda)$  which sends a  $k$ -graph  $\Lambda$  to its skeleton is reversible: given a skeleton  $(E, F)$ , there is a unique  $k$ -graph  $\Lambda_{(E, F)}$  such that  $(E_{\Lambda_{(E, F)}}, F_{\Lambda_{(E, F)}}) = (E, F)$ .

Note that if  $k = 1$ , then there are no factorisation rules to list, so every (1-coloured) directed graph specifies a 1-graph. Likewise, if  $k = 2$ , then (2) above is trivial because we cannot have distinct  $i, j, l \in \{1, \dots, k\}$ , so every bi-coloured graph together with factorisation rules satisfying (1) specifies a 2-graph.

### 3. CROSSED PRODUCTS BY $\mathbb{Z}^l$ AS HIGHER-RANK GRAPH ALGEBRAS

In this section we show how an action  $\alpha$  of  $\mathbb{Z}^l$  on a finitely aligned  $k$ -graph  $\Lambda$  induces an action  $\tilde{\alpha}$  of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$ , generalising the assertion of [10, page 16] to the finitely aligned setting. We then show how to realise the crossed-product  $C^*$ -algebra  $C^*(\Lambda) \rtimes_{\tilde{\alpha}} \mathbb{Z}^l$  as the universal algebra of a  $(k + l)$ -graph  $\Lambda \times_{\alpha} \mathbb{Z}^l$ .

**Proposition 3.1.** *Let  $\Lambda$  be a finitely aligned  $k$ -graph and let  $s_\Lambda$  be the universal Cuntz-Krieger  $\Lambda$ -family in  $C^*(\Lambda)$ .*

- (1) *Let  $\phi$  be an automorphism of  $\Lambda$ . Then there is a unique automorphism  $\tilde{\phi}$  of  $C^*(\Lambda)$  satisfying  $\tilde{\phi}(s_\lambda) = s_{\phi(\lambda)}$  for all  $\lambda \in \Lambda$ .*
- (2) *Let  $G$  be a group, and suppose that  $g \mapsto \alpha_g$  is an action of  $G$  on  $\Lambda$  by automorphisms. Then  $g \mapsto \tilde{\alpha}_g$  is an action of  $G$  on  $C^*(\Lambda)$  by automorphisms.*

*Proof.* 1) It is easy to check that a graph isomorphism preserves minimal common extensions and finite exhaustive sets. It follows from this that there is a Cuntz-Krieger  $\Lambda$ -family  $t_\Lambda^\phi$  defined by  $t_\lambda^\phi := s_{\phi(\lambda)}$  for  $\lambda \in \Lambda$ . The universal property of  $C^*(\Lambda)$  therefore furnishes us with a  $C^*$ -homomorphism  $\tilde{\phi}$  of  $C^*(\Lambda)$  satisfying  $\tilde{\phi}(s_\lambda) = t_\lambda^\phi = s_{\phi(\lambda)}$  for all  $\lambda \in \Lambda$ . Applying the same argument to  $\phi^{-1} \in \text{Aut}(\Lambda)$  gives another  $C^*$ -homomorphism  $\widetilde{\phi^{-1}} : C^*(\Lambda) \rightarrow C^*(\Lambda)$ , and since  $\widetilde{\phi^{-1}} \circ \tilde{\phi}$  fixes all the generators of  $C^*(\Lambda)$ ,  $\widetilde{\phi^{-1}}$  is an inverse for  $\tilde{\phi}$  and in particular,  $\tilde{\phi}$  is an automorphism.

2) Let  $1_G$  denote the identity element of  $G$ . Since  $\alpha$  is an action,

$$\tilde{\alpha}_{1_G}(s_\lambda) = s_\lambda, \quad \tilde{\alpha}_{g^{-1}}(\tilde{\alpha}_g(s_\lambda)) = s_\lambda \quad \text{and} \quad \tilde{\alpha}_g(\tilde{\alpha}_h(s_\lambda)) = \tilde{\alpha}_{gh}(s_\lambda),$$

for each generator  $s_\lambda$  of  $C^*(\Lambda)$  and all  $g, h \in G$ . It follows that  $\tilde{\alpha}$  is an action of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$  by automorphisms.  $\square$

We now show how to construct a  $(k + l)$ -graph  $\Lambda \times_{\alpha} \mathbb{Z}^l$  from an action of  $\mathbb{Z}^l$  on a  $k$ -graph  $\Lambda$ . We show in Theorem 3.5 that the  $C^*$ -algebra of this  $(k + l)$ -graph is isomorphic to the crossed-product  $C^*(\Lambda) \rtimes_{\tilde{\alpha}} \mathbb{Z}^l$ .

**Proposition 3.2.** *Let  $\Lambda$  be a  $k$ -graph, and suppose that  $\alpha$  is an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. Then there is a unique  $(k + l)$ -graph  $\Lambda \times_{\alpha} \mathbb{Z}^l$  such that*

- (1)  $(\Lambda \times_{\alpha} \mathbb{Z}^l)^{(p, m)} = \Lambda^p \times \{m\}$  for all  $(p, m) \in \mathbb{N}^{k+l}$ ;

- (2)  $r(\lambda, m) = (r(\lambda), 0_l)$  and  $s(\lambda, m) = (\alpha_{-m}(s(\lambda)), 0_l)$  for all  $\lambda \in \Lambda$  and  $m \in \mathbb{N}^l$ ; and
- (3)  $(\mu, m)(\nu, n) = (\mu\alpha_m(\nu), m+n)$  whenever  $s(\mu, m) = r(\nu, n)$ .

Moreover,  $\Lambda \times_\alpha \mathbb{Z}^l$  is finitely aligned if and only if  $\Lambda$  is finitely aligned,  $\Lambda \times_\alpha \mathbb{Z}^l$  is row-finite if and only if  $\Lambda$  is row-finite, and  $\Lambda \times_\alpha \mathbb{Z}^l$  has no sources if and only if  $\Lambda$  has no sources.

*Remark 3.3.* Although the notation may suggest otherwise, the  $k$ -graph  $\Lambda \times_\alpha \mathbb{Z}^l$  is equal as a set to  $\Lambda \times \mathbb{N}^l$  rather than to  $\Lambda \times \mathbb{Z}^l$ .

*Proof of Proposition 3.2.* The details of this proof are quite messy, but the idea is straightforward; we present only the idea here.

The discussion in Section 2.6 shows that it suffices to produce the skeleton of  $\Lambda \times_\alpha \mathbb{Z}^l$ . That is, a  $(k+l)$ -coloured graph  $E$  and an allowable list  $F$  of factorisation rules. To obtain  $E$ , we begin with a copy  $E_\Lambda \times \{0_l\}$  of the  $k$ -coloured graph associated to  $\Lambda$  and augment it as follows: for each  $1 \leq i \leq l$  and each  $v \in \Lambda^{0_k}$ , we add an edge  $(v, e_i)$  to  $E_{k+i}^1$  directed from the vertex  $(\alpha_{e_i}^{-1}(v), 0_l)$  to the vertex  $(v, 0_l)$ . The factorisation rules  $F$  are specified as follows.

- If  $f_1 f_2 = f'_2 f'_1$  in  $\Lambda$ , then  $(f_1, 0_l)(f_2, 0_l) = (f'_2, 0_l)(f'_1, 0_l)$  belongs to  $F$ ; that is, the factorisation rules amongst edges from  $E_\Lambda$  are unchanged.
- For  $f \in \Lambda^{e_j}$  and  $1 \leq i \leq l$ ,  $(f, 0_l)(s(f), e_i) = (r(f), e_i)(\alpha_{e_i}^{-1}(f), 0_l)$ .
- For  $i \neq j$  in  $\{1, \dots, l\}$  and  $v \in \Lambda^{0_k}$ ,  $(v, e_i)(\alpha_{e_i}^{-1}(v), e_j) = (v, e_j)(\alpha_{e_j}^{-1}(v), e_i)$ .

We must check that this collection  $F$  satisfies conditions (1) and (2) of Section 2.6. That  $\alpha$  is an action ensures that the  $\alpha_{e_i}$  commute, so Condition (1) is satisfied for  $k+1 \leq i < j \leq k+l$ . That each  $\alpha_{e_i}$  is an automorphism guarantees that Condition (1) is satisfied for  $1 \leq i \leq k$  and  $1 \leq j \leq l$ . Condition (1) is satisfied for  $1 \leq i < j \leq k$  because Condition (1) is satisfied in  $E_\Lambda$ . Associativity of composition in  $\Lambda$  and that  $\alpha$  is an action ensure that the above factorisation rules satisfy Condition (2).

It follows from [6, Example 1.5(4) and Theorems 2.1 and 2.2] that there is a unique  $(k+l)$ -graph  $\Lambda \times_\alpha \mathbb{Z}^l$  with skeleton  $(E, F)$ . Fix  $(p, m) \in \mathbb{N}^{k+l}$ , and  $\lambda \in \Lambda^p$ . Factorise  $\lambda = f_1 \dots f_{|p|}$  as a sequence of edges from  $E_\Lambda$ . Let  $|m|$  denote the length  $m_1 + \dots + m_k$  of  $m$  as an element of  $\mathbb{N}^l$  with respect to the usual basis, and fix  $a_1, \dots, a_{|m|} \in \{1, \dots, l\}$  such that  $m = e_{a_1} + e_{a_2} + \dots + e_{a_{|m|}}$ . This gives us a path

$$f_1 \dots f_{|p|}(s(\lambda), a_1)(\alpha_{a_1}^{-1}(s(\lambda)), a_2) \dots (\alpha_{m-a_{|m|}}^{-1}(s(\lambda)), a_{|m|})$$

in  $E$ , and hence a path  $\xi(\lambda, m) \in (\Lambda \times_\alpha \mathbb{Z}^l)^{(p, m)}$ . One checks using the definition of  $F$  that  $\xi(\lambda, m)$  does not depend on the choice of factorisation of  $\lambda$  into edges or the decomposition of  $m$  into generators. Using the factorisation property in  $\Lambda \times_\alpha \mathbb{Z}^l$ , one checks that every path in  $(\Lambda \times_\alpha \mathbb{Z}^l)^{(p, m)}$  is of the form  $\xi(\lambda, m)$  for some  $\lambda \in \Lambda^p$ . It follows that  $\xi$  is a bijection between  $\Lambda^p \times \{m\}$  and  $(\Lambda \times_\alpha \mathbb{Z}^l)^{(p, m)}$ . One then checks using the definition of  $(E, F)$  that this bijection satisfies (2) and (3).

To prove the final statement of the Proposition, we first claim that for  $\mu, \nu \in \Lambda$  and  $m, n \in \mathbb{N}^l$ ,

$$(3.1) \quad \text{MCE}_{\Lambda \times_{\alpha} \mathbb{Z}^l}((\mu, m), (\nu, n)) = \text{MCE}_{\Lambda}(\mu, \nu) \times \{m \vee n\}.$$

To see this, suppose first that  $(\lambda, p) \in \text{MCE}_{\Lambda \times_{\alpha} \mathbb{Z}^l}((\mu, m), (\nu, n))$ . Then  $d(\lambda, p) = (d(\mu) \vee d(\nu), m \vee n)$ , which gives  $p = m \vee n$  and  $d(\lambda) = d(\mu) \vee d(\nu)$ . Moreover,  $(\lambda, p) = (\mu, m)(\eta, p - m)$  where  $\mu\alpha_m(\eta) = \lambda$ . Likewise,  $(\lambda, p) = (\nu, n)(\xi, p - n)$  where  $\nu\alpha_n(\xi) = \lambda$ . Hence  $\lambda \in \text{MCE}_{\Lambda}(\mu, \nu)$  and  $(\lambda, p) \in \text{MCE}_{\Lambda}(\mu, \nu) \times \{m \vee n\}$ .

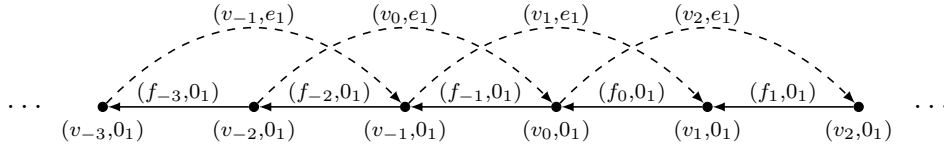
Now suppose that  $\lambda \in \text{MCE}_{\Lambda}(\mu, \nu)$ , and let  $p := m \vee n$ . By definition, we have  $d(\lambda, p) = d(\mu, m) \vee d(\nu, n)$ . Write  $\lambda = \mu\eta = \nu\xi$ . By definition of composition in  $\Lambda \times_{\alpha} \mathbb{Z}^l$  we have  $(\lambda, p) = (\mu, m)(\alpha_m^{-1}(\eta), p - m) = (\nu, n)(\alpha_n^{-1}(\xi), p - n)$ , so  $(\lambda, p) \in \text{MCE}_{\Lambda \times_{\alpha} \mathbb{Z}^l}((\mu, m), (\nu, n))$ . This establishes (3.1), and in particular implies that  $\Lambda \times_{\alpha} \mathbb{Z}^l$  is finitely aligned if and only if  $\Lambda$  is finitely aligned.

Next note that by construction of  $\Lambda \times_{\alpha} \mathbb{Z}^l$ ,

$$|v\Lambda^p| = |(v, 0_l)(\Lambda \times_{\alpha} \mathbb{Z}^l)^{(p, m)}| \quad \text{for all } v \in \Lambda^0 \text{ and } (p, m) \in \mathbb{N}^{k+l}.$$

In particular,  $\Lambda \times_{\alpha} \mathbb{Z}^l$  is row-finite if and only if  $\Lambda$  is row-finite, and  $\Lambda \times_{\alpha} \mathbb{Z}^l$  has no sources if and only if  $\Lambda$  has no sources.  $\square$

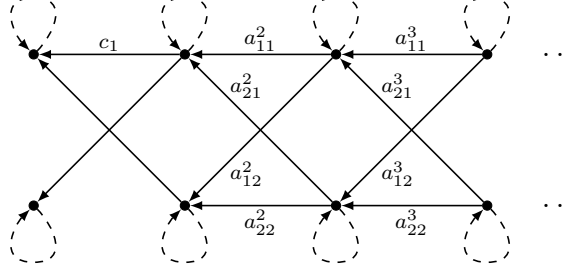
*Examples 3.4.* (1) Let  $k = l = 1$ , and let  $\Lambda$  be the 1-graph consisting of a two-sided infinite path with vertices  $\{v_n : n \in \mathbb{Z}\}$  and edges  $\{f_n : n \in \mathbb{Z}\}$  where  $r(f_n) = v_n$  and  $s(f_n) = v_{n+1}$ . Let  $\alpha$  be the automorphism of  $\Lambda$  satisfying  $\alpha(v_n) = v_{n+2}$  and  $\alpha(f_n) = f_{n+2}$ . Then the skeleton of  $\Lambda \times_{\alpha} \mathbb{Z}^l$  is as follows:



- (2) Fix  $\theta \in [0, 1] \setminus \mathbb{Q}$ , and let  $[c_1, c_2, \dots]$  be its reduced continued fraction expansion. For each  $n \in \mathbb{N}$ , let  $\Phi_n$  be the matrix  $\begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}$ . Let  $T_n$  be the sequence of triangular numbers  $T_n := \sum_{i=1}^n i = \frac{n(n+1)}{2}$ , and for  $n \in \mathbb{N}$  let  $A_n$  denote the matrix  $\prod_{i=T_{n-1}+1}^{T_n} \Phi_i$ . Let  $a_{i,j}^n$  denote the  $i, j$ -entry of  $A_n$  for  $i, j = 1, 2$ . Let  $E$  be the directed graph with vertices  $\{v_i^m : m \in \mathbb{N}, i \in \{1, 2\}\}$ , and with edges  $\{e_{i,j}^m(n) : m \in \mathbb{N}, i, j \in \{1, 2\}, n \in \mathbb{Z}/a_{i,j}^n\mathbb{Z}\}$ , where  $r(e_{i,j}^m(n)) = v_i^m$  and  $s(e_{i,j}^m(n)) = v_j^{m+1}$ . So  $E$  consists of the vertices and solid edges in the diagram below (where a label  $n$  on an arrow indicates a bundle of  $n$  parallel



edges).



Define an automorphism  $\alpha_1$  of  $E$  as follows:  $\alpha_1(v_i^m) = v_i^m$  for all  $v \in E^0$ , and  $\alpha_1(e_{ij}^m(n)) := e_{ij}^m(n+1)$ . That is,  $\alpha_1$  fixes all the vertices, and cyclicly permutes parallel edges. Let  $\Lambda = E^*$  be the path-category of  $E$  regarded as a 1-graph. Then  $\alpha_1$  extends uniquely to an automorphism  $\bar{\alpha}_1$  of  $\Lambda$ . Let  $\bar{\alpha}$  be the action of  $\mathbb{Z}$  on  $\Lambda$  generated by  $\bar{\alpha}_1$ . The skeleton of  $\Lambda \times_{\bar{\alpha}} \mathbb{Z}$  is the 2-coloured graph pictured above, which is identical to the one in [15, Figure 3]. In particular, it follows from [15, Example 6.5] that the  $C^*$ -algebra of this 2-graph is Morita equivalent to the irrational rotation algebra  $A_\theta$ .

**Theorem 3.5.** *Let  $\Lambda$  be a finitely aligned  $k$ -graph, and  $\alpha$  an action of  $\mathbb{Z}^l$  by automorphisms of  $\Lambda$ . Let  $\tilde{\alpha}$  be the corresponding action of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$  as in Proposition 3.1, and denote by  $\pi : C^*(\Lambda) \rightarrow C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  and by  $U : \mathbb{Z}^l \rightarrow \mathcal{M}(C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l)$  the universal generating covariant representation of the dynamical system  $(C^*(\Lambda), \mathbb{Z}^l, \tilde{\alpha})$ . There is a unique isomorphism  $\phi : C^*(\Lambda \times_{\alpha} \mathbb{Z}^l) \rightarrow C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  which satisfies  $\phi(s_{(\lambda, m)}) := \pi(s_\lambda)U_m$  for each  $\lambda \in \Lambda$  and  $m \in \mathbb{N}^l$ .*

*Proof.* For  $(\lambda, m) \in \Lambda \times_{\alpha} \mathbb{Z}^l$ , let  $t_{(\lambda, m)} := \pi(s_\lambda)U_m$ . We claim that  $\{t_{(\lambda, m)} : (\lambda, m) \in \Lambda \times_{\alpha} \mathbb{Z}^l\}$  is a Cuntz-Krieger  $\Lambda \times_{\alpha} \mathbb{Z}^l$ -family. First note that  $U_{0_l}$  is the identity element of  $\mathcal{M}(C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l)$ , so  $t_{(\lambda, 0_l)} = \pi(s_\lambda)$  for each  $\lambda \in \Lambda$ . Since  $\pi$  is an isomorphism, it follows that  $\{t_{(\lambda, 0_l)} : \lambda \in \Lambda\}$  forms a Cuntz-Krieger  $\Lambda$ -family in  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$ . In particular, the elements  $\{t_{(v, 0_l)} : (v, 0_l) \in (\Lambda \times_{\alpha} \mathbb{Z}^l)^{0_{k+l}}\}$  are mutually orthogonal projections, which establishes (TCK1). For  $(\mu, m)$  and  $(\nu, n)$  with  $r(\nu) = \alpha_m^{-1}(s(\mu))$ , we have

$$\begin{aligned}
 t_{(\mu, m)}t_{(\nu, n)} &= \pi(s_\mu)U_m\pi(s_\nu)U_n \\
 &= \pi(s_\mu)U_m\pi(s_\nu)U_m^*U_mU_n \\
 &= \pi(s_\mu)\pi(\tilde{\alpha}_m(s_\nu))U_mU_n \\
 &= \pi(s_{\mu\alpha_m(\nu)})U_{m+n} \\
 &= t_{(\mu, m)(\nu, n)}.
 \end{aligned}$$

This establishes (TCK2).

To show that (TCK3) holds, fix  $(\mu, m), (\nu, n) \in \Lambda \times_\alpha \mathbb{Z}^l$  with  $r(\mu, m) = r(\nu, n)$ . We calculate:

$$\begin{aligned} t_{(\mu, m)} t_{(\mu, m)}^* t_{(\nu, n)} t_{(\nu, n)}^* &= \pi(s_\mu) U_m U_m^* \pi(s_\mu)^* \pi(s_\nu) U_n U_n^* \pi(s_\nu)^* \\ &= \pi \left( \sum_{\lambda \in \text{MCE}(\mu, \nu)} s_\lambda s_\lambda^* \right) \end{aligned}$$

because  $U_m, U_n$  are unitaries and  $\{\pi(s_\lambda) : \lambda \in \Lambda\}$  is a Cuntz-Krieger  $\Lambda$ -family. Hence (3.1) implies that

$$t_{(\mu, m)} t_{(\mu, m)}^* t_{(\nu, n)} t_{(\nu, n)}^* = \sum_{(\lambda, p) \in \text{MCE}_{\Lambda \times_\alpha \mathbb{Z}^l}((\mu, m), (\nu, n))} t_{(\lambda, p)} t_{(\lambda, p)}^*,$$

and multiplying both sides of this equation on the left by  $t_{(\mu, m)}^*$  and on the right by  $t_{(\nu, n)}$  gives (TCK3).

To show that (CK) holds, we first establish that finite exhaustive sets in  $\Lambda \times_\alpha \mathbb{Z}^l$  project onto finite exhaustive sets in  $\Lambda$ . Fix a vertex  $(v, 0_l)$  of  $\Lambda \times_\alpha \mathbb{Z}^l$ , and let  $F$  be an exhaustive subset of  $(v, 0_l) \Lambda^\alpha$ . Let  $F_1 := \{\lambda \in \Lambda : \text{there exists } m \in \mathbb{N}^l \text{ such that } (\lambda, m) \in F\}$ , so that  $F_1$  is the projection of  $F$  onto  $\Lambda$ . We claim that  $F_1$  is exhaustive in  $\Lambda$ . To see this, fix  $\mu \in \Lambda$  with  $r(\mu) = v$ . Then  $(\mu, 0_l) \in (v, 0_l) (\Lambda \times_\alpha \mathbb{Z}^l)$ . Hence there exists  $(\lambda, p) \in F$  with  $\text{MCE}_{\Lambda \times_\alpha \mathbb{Z}^l}((\lambda, p), (\mu, 0_l)) \neq \emptyset$ . Equation (3.1) implies that  $\text{MCE}_{\Lambda \times_\alpha \mathbb{Z}^l}((\lambda, p), (\mu, 0_l)) = \text{MCE}_\Lambda(\lambda, \mu) \times \{p\}$ , so  $\text{MCE}_\Lambda(\lambda, \mu) \neq \emptyset$ . Since  $\mu \in v\Lambda$  was arbitrary, and since  $\lambda \in F_1$ , it follows that  $F_1$  is exhaustive. Now to establish (CK), suppose that  $F$  is finite exhaustive in  $\Lambda \times_\alpha \mathbb{Z}^l$ , so that, by the above,  $F_1$  is finite exhaustive in  $\Lambda$ . Then

$$\begin{aligned} \prod_{(\lambda, p) \in F} (t_{(v, 0_l)} - t_{(\lambda, p)} t_{(\lambda, p)}^*) &= \prod_{(\lambda, p) \in F} (\pi(s_v) U_0 - \pi(s_\lambda) U_p U_p^* \pi(s_\lambda)^*) \\ &= \pi \left( \prod_{\lambda \in F_1} (s_v - s_\lambda s_\lambda^*) \right) \\ &= 0 \end{aligned}$$

because  $s_\Lambda$  satisfies relation (CK).

We have now proved that  $t_{\Lambda \times_\alpha \mathbb{Z}^l}$  is a Cuntz-Krieger  $(\Lambda \times_\alpha \mathbb{Z}^l)$ -family.

The universal property of  $C^*(\Lambda \times_\alpha \mathbb{Z}^l)$  implies that there is a homomorphism  $\phi : C^*(\Lambda \times_\alpha \mathbb{Z}^l) \rightarrow C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  which satisfies  $\phi(s_{(\lambda, m)}) = \pi(s_\lambda) U_m$  for all  $\lambda \in \Lambda, m \in \mathbb{N}^l$ .

We claim that  $\phi$  is surjective. The crossed product  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  is, by definition, generated by elements of the form  $\pi(a) U_m$  where  $a \in C^*(\Lambda)$  and  $m \in \mathbb{Z}^k$ . Hence it suffices to show that  $\pi(a) U_m$  is in the image of  $\phi$  for each  $a \in C^*(\Lambda)$  and  $m \in \mathbb{Z}^l$ . Since  $C^*(\Lambda) = \overline{\text{span}} \{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$ , it therefore suffices to show that  $s_\mu s_\nu^* U_m$  is in the range of  $\phi$  for each  $m \in \mathbb{Z}^l$  and  $\mu, \nu \in \Lambda$ . Fix  $\mu, \nu \in \Lambda$  and  $m \in \mathbb{Z}^l$ . Write  $m = m_+ - m_-$  where  $m_+, m_- \in \mathbb{N}^l$ . Since  $U_m^* s_\nu^* U_m = \tilde{\alpha}_{-m}(s_\nu^*) = s_{\alpha_{-m}(\nu)}^*$ , we have

$$s_\mu s_\nu^* U_m = s_\mu U_m U_m^* s_\nu^* U_m = s_\mu U_{m_+} U_{m_-}^* s_{\alpha_{-m}(\nu)}^* = \phi(t_{(\mu, m_+)} t_{(\alpha_{-m}(\nu), m_-)}^*).$$

Hence  $\phi$  is surjective. It remains only to show that  $\phi$  is injective.

Let  $\gamma$  denote the gauge action of  $\mathbb{T}^k$  on  $C^*(\Lambda)$  and let  $\gamma^\alpha$  denote the gauge action of  $\mathbb{T}^{k+l}$  on  $C^*(\Lambda \times_\alpha \mathbb{Z}^l)$ . The universal property of the crossed product  $C^*(\Lambda) \times_{\hat{\alpha}} \mathbb{Z}^l$  can be used to deduce that there is an action  $\bar{\gamma}$  of  $\mathbb{T}^k$  on  $C^*(\Lambda) \times_{\hat{\alpha}} \mathbb{Z}^l$  which satisfies  $\bar{\gamma}_z(\pi(a)U_m) = \pi(\gamma_z(a))U_m$  for all  $z \in \mathbb{T}^k$ ,  $a \in C^*(\Lambda)$ , and  $m \in \mathbb{Z}^l$ . Let  $\hat{\alpha}$  denote the dual action of  $\mathbb{T}^l = \widehat{\mathbb{Z}^l}$  on  $C^*(\Lambda) \times_{\hat{\alpha}} \mathbb{Z}^l$  which satisfies  $\hat{\alpha}_w(\pi(a)U_m) := w^m \pi(a)U_m$ . Identifying  $\mathbb{T}^{k+l}$  with  $\{(z, w) : z \in \mathbb{T}^k, w \in \mathbb{T}^l\}$ , define automorphisms  $\{\beta_{(z, w)} : (z, w) \in \mathbb{T}^{k+l}\}$  of  $C^*(\Lambda) \times_{\hat{\alpha}} \mathbb{Z}^l$  by  $\beta_{(z, w)} := \bar{\gamma}_z \circ \hat{\alpha}_w$ . It is easy to see that  $\beta$  determines an action of  $\mathbb{T}^{k+l}$  on the crossed product algebra, and one can check on generators that  $\phi \circ \gamma_{(z, w)}^\alpha = \beta_{(z, w)} \circ \phi$  for all  $(z, w) \in \mathbb{T}^{k+l}$ . Since  $\pi$  is injective, we have  $t_v = \pi(s_v) \neq 0$  for all  $v$ . The gauge-invariant uniqueness theorem [19, Theorem 4.2] therefore implies that  $\phi$  is injective.  $\square$

**3.1. Recognising crossed-product  $k$ -graphs.** There is a converse of sorts to Proposition 3.2; that is, one can tell by looking at a  $(k+l)$ -graph whether or not it is of the form  $\Lambda \times_\alpha \mathbb{Z}^l$  for some  $k$ -graph  $\Lambda$  and some action  $\alpha$  of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms.

We require some notation. Given a  $(k+l)$ -graph  $\Xi$ , we write  $\Xi^{(\mathbb{N}^k, 0_l)}$  for the  $k$ -graph with morphisms  $\bigcup_{p \in \mathbb{N}^k} \Xi^{(p, 0_l)}$  and degree functor  $d^{(\mathbb{N}^k, 0_l)}(\xi) = (d(\xi)_1, \dots, d(\xi)_k) \in \mathbb{N}^k$ .

**Proposition 3.6.** *Let  $\Xi$  be a  $(k+l)$ -graph. Suppose that for every  $v \in \Xi^{0_{k+l}}$  and every  $j \in \{1, \dots, l\}$ , we have  $|v\Xi^{(0_k, e_j)}| = |\Xi^{(0_k, e_j)}v| = 1$ . Then for each vertex  $v \in \Xi^{0_{k+l}}$  and  $m \in \mathbb{N}^l$  there is a unique path  $\eta_{v, m}$  in  $\Xi^{(0_k, m)}v$ , and there is a unique action  $\alpha$  of  $\mathbb{Z}^l$  on  $\Xi^{(\mathbb{N}^k, 0_l)}$  satisfying  $\alpha_m(\xi) := (\eta_{r(\xi), m}\xi)(0_{k+l}, d(\xi))$  for all  $\xi \in \Xi^{(\mathbb{N}^k, 0_l)}$  and  $m \in \mathbb{N}^l$ . Moreover,  $\Xi$  is isomorphic to  $\Xi^{(\mathbb{N}^k, 0_l)} \times_\alpha \mathbb{Z}^l$ .*

*Proof.* Since  $|\Xi^{(0_k, e_j)}v| = 1$  for all  $v$  and  $j$ , it is immediate from the factorisation property that  $|\Xi^{(0_k, m)}v| = 1$  for all  $m \in \mathbb{N}^l$  and  $v \in \Xi^{0_{k+l}}$ . Arguments like those of [15, Lemma 3.3] show that for  $m \in \mathbb{N}^l$ , the formula

$$\xi \mapsto (\eta_{r(\xi), m}\xi)(0_{k+l}, d(\xi))$$

determines an automorphism  $\alpha_m$  of  $\Xi^{(\mathbb{N}^k, 0_l)}$  for each  $m \in \mathbb{N}^l$  and that  $\alpha_m \circ \alpha_n = \alpha_{m+n}$  for all  $m, n \in \mathbb{N}^l$ . Hence the  $\alpha_m$  determine an action  $\alpha$  of  $\mathbb{Z}^l$  on  $\Xi^{(\mathbb{N}^k, 0_l)}$  as claimed: writing  $m \in \mathbb{Z}^l$  as  $m = m_+ - m_-$  where  $m_+, m_- \in \mathbb{N}^l$ , we define  $\alpha_m := \alpha_{m_+} \circ \alpha_{m_-}^{-1}$ . It is easy to check that the skeleton of  $\Xi^{(\mathbb{N}^k, 0_l)} \times_\alpha \mathbb{Z}^l$  is identical to that of  $\Xi$ , so the isomorphism  $\Xi \cong \Xi^{(\mathbb{N}^k, 0_l)} \times_\alpha \mathbb{Z}^l$  follows from the uniqueness assertion of [6, Theorem 2.1].  $\square$

*Examples 3.7.* (1) As in Section 3 of [15], let  $\Lambda$  be a row-finite 2-graph with no sources such that  $\Lambda^{(\mathbb{N}, 0)}$  contains no cycles and each vertex  $v \in \Lambda^{0_2}$  is the range of an isolated cycle in  $\Lambda^{(0, \mathbb{N})}$ . It was shown in [15, Theorem 3.1],  $C^*(\Lambda)$  is an AT-algebra.

By Proposition 3.6, we see that  $C^*(\Lambda)$  is isomorphic to a crossed product by  $\mathbb{Z}$  of the AF algebra  $C^*(\Lambda^{(\mathbb{N},0)})$ . In Section 6 we show how to use the Pimsner-Voiculescu exact sequence to calculate the  $K$ -theory of  $C^*(\Lambda)$ .

(2) Consider the 3-graphs  $\varprojlim(\Delta_2/H_n, p_n)$  discussed in [12, Section 6.4]. Since each  $\Delta_2/H_n$  is a quotient of  $\Delta_2$ , each vertex in  $\varprojlim(\Delta_2/H_n, p_n)$  both emits and receives exactly one edge of degree  $e_1$  and exactly one edge of degree  $e_2$ . After a change of basis for  $\mathbb{N}^3$ , we can therefore use Proposition 3.6 and Theorem 3.5 to realise  $C^*(\varprojlim(\Delta_2/H_n, p_n))$  as a crossed product of the AF algebra  $C^*(\varprojlim(\Delta_2/H_n, p_n)^{(0_2, \mathbb{N})})$  by  $\mathbb{Z}^2$ . Indeed, taking the corner generated by the lone vertex in  $\Delta_2/H_1$ , we recover the crossed product of  $C_0(\varprojlim \mathbb{Z}^2/H_n)$  by a generalised odometer action discussed in [12, Remark 6.12].

(3) Fix an integer  $l \geq 1$ . It is easy to verify that there is an isomorphism of  $C^*(\Delta_l)$  onto  $\mathcal{K}(\ell^2(\mathbb{Z}^l))$  which takes  $s_{(m, m+n)}$  to the matrix unit  $\theta_{m, m+n}$ ; we will henceforth identify  $C^*(\Delta_l)$  with  $\mathcal{K}(\ell^2(\mathbb{Z}^l))$  via this isomorphism.

Fix a  $k$ -graph  $\Lambda$ . Consider the Cartesian product  $(k+l)$ -graph  $\Lambda \times \Delta_l := \{(\lambda, (m, n)) : \lambda \in \Lambda, (m, n) \in \Delta_l\}$  with coordinatewise range, source and composition maps, and degree map  $d(\lambda, (m, n)) = (d(\lambda), d(m, n)) = (d(\lambda), n - m)$  (see [10, Proposition 1.8]). Clearly each vertex of  $\Lambda \times \Delta_l$  emits and receives exactly one edge of degree  $e_j$  for  $k+1 \leq j \leq k+l$ . Moreover,  $\Lambda^{(\mathbb{N}^k, 0_l)} \cong \bigsqcup_{m \in \mathbb{Z}^l} \Lambda \times \{m\}$  is a disjoint union of copies of  $\Lambda$  indexed by  $\mathbb{Z}^l$ , and the action  $\alpha$  on this  $k$ -graph arising from Proposition 3.6 is implemented by translation in the  $\mathbb{Z}^l$  coordinate; that is,  $\alpha'_m(\lambda, (m, n)) = (\lambda, (m + m', n + m'))$ . Hence,

$$C^*(\Lambda^{(\mathbb{N}^k, 0_l)}) \cong \bigoplus_{z \in \mathbb{Z}^l} C^*(\Lambda) \cong C^*(\Lambda) \otimes c_0(\mathbb{Z}^l),$$

and under this identification,  $\tilde{\alpha}$  becomes  $\text{id} \otimes \text{lt}$ . Since  $c_0(\mathbb{Z}^l) \times_{\text{lt}} \mathbb{Z}^l$  is canonically isomorphic to  $\mathcal{K}(\ell^2(\mathbb{Z}^l))$ , and since  $C^*(\Delta_l)$  is also canonically isomorphic to  $\mathcal{K}(\ell^2(\mathbb{Z}^l))$  Theorem 3.5 re-proves the isomorphism  $C^*(\Lambda \times \Delta_l) \cong C^*(\Lambda) \otimes C^*(\Delta_l)$  obtained from [10, Corollary 3.5(iv)].

**3.2. Takai duality.** In this section we show how our construction together with the skew-product construction of [10, 13] provides a graph-theoretic realisation of Takai duality for the  $\mathbb{Z}^l$  actions discussed in Theorem 3.5.

We require some background regarding the skew-product of a  $k$ -graph by an abelian group  $G$  and its relationship to a crossed product by an action of the dual group  $\widehat{G}$  (see [10, Section 5]). This construction has been generalised to nonabelian groups  $G$  using coactions (see [13]), but for our purposes, the generality of [10] suffices.

Let  $\Xi$  be a  $k$ -graph, and let  $c : \Xi \rightarrow G$  be a cocycle into an abelian group  $(G, +)$ ; that is,  $c(\mu\nu) = c(\mu) + c(\nu)$  whenever  $s(\mu) = r(\nu)$ . Following the conventions of [13], we define the *skew-product  $k$ -graph*  $\Xi \times_c G$  to be equal as a set to  $\Xi \times G$ , with

structure maps

$$\begin{aligned} s_c(\mu, g) &:= (s(\mu), g) & r_c(\mu, g) &= (r(\mu), c(\mu) + g) \\ (\mu, c(\nu) + g)(\nu, g) &:= (\mu\nu, g) & \text{and} & & d_c(\mu, g) &:= d(\mu) \end{aligned}$$

for all  $\mu, \nu \in \Xi$  such that  $s(\mu) = r(\nu)$ , and all  $g \in G$ .

There is an action  $\delta^c$  of  $\widehat{G}$  on  $C^*(\Xi)$  satisfying  $\delta_\phi^c(s_\mu) := \phi(c(\mu))s_\mu$  for all  $\phi \in \widehat{G}$  and  $\mu \in \Xi$ . Corollary 5.3 of [10] states that  $C^*(\Xi \times_c G) \cong C^*(\Xi) \times_{\delta^c} \widehat{G}$ .

Given an action  $\alpha$  of  $\mathbb{Z}^l$  on a  $k$ -graph  $\Lambda$ , the map  $c : \Lambda \times_\alpha \mathbb{Z}^l \rightarrow \mathbb{Z}^l$  defined by  $c(\lambda, m) := -m$  for all  $\lambda \in \Lambda$  and  $m \in \mathbb{N}^l$  is a cocycle (see Proposition 3.2). We may therefore form the skew-product  $(k+l)$ -graph  $(\Lambda \times_\alpha \mathbb{Z}^l) \times_c \mathbb{Z}^l$ .

Finally, recall from [10, Proposition 1.8] that given a  $k$ -graph  $\Lambda$  and an  $l$ -graph  $\Gamma$ , the cartesian product  $\Lambda \times \Gamma$  becomes a  $(k+l)$ -graph with structure maps and degree functor defined coordinatewise.

**Theorem 3.8.** *Fix a row-finite  $k$ -graph  $\Lambda$  with no sources and an integer  $l \geq 1$ . Let  $\alpha$  be an action of  $\mathbb{Z}^l$  on  $\Lambda$ . Then the formula*

$$\rho((\lambda, m), n) := (\alpha_{n-m}(\lambda), (n - m, n))$$

*determines an isomorphism of the skew-product graph  $(\Lambda \times_\alpha \mathbb{Z}^l) \times_c \mathbb{Z}^l$  onto the cartesian product  $\Lambda \times \Delta_l$ .*

*Proof.* To establish that  $\rho$  is an isomorphism of  $(k+l)$ -graphs, first observe that it is bijective and degree-preserving by definition. We therefore need only show that it preserves range, source and composition. We have

$$\begin{aligned} r(\rho((\lambda, m), n)) &= r(\alpha_{n-m}(\lambda), (n - m, n)) = (\alpha_{n-m}(r(\lambda)), (n - m, n - m)) \\ &= \rho((r(\lambda), 0_l), n - m) = \rho(r_c((\lambda, m), n)) \end{aligned}$$

and

$$\begin{aligned} s(\rho((\lambda, m), n)) &= s(\alpha_{n-m}(\lambda), (n - m, n)) = (\alpha_{n-m}(s(\lambda)), (n, n)) \\ &= \rho((\alpha_{-m}(s(\lambda)), 0_l), n) = \rho(s_c((\lambda, m), n)), \end{aligned}$$

establishing that  $\rho$  preserves the range and source maps. To see that  $\rho$  preserves composition, fix  $\mu, \nu \in \Lambda$  and  $m, n, m' \in \mathbb{N}^l$  such that  $s(\mu) = \alpha_{m'}(r(\nu))$ . Then  $((\mu, m'), n - m)$  and  $((\nu, m), n)$  are composable in  $(\Lambda \times_\alpha \mathbb{Z}^l) \times_c \mathbb{Z}^l$  with

$$((\mu, m'), n - m)((\nu, m), n) = ((\mu\alpha_{m'}(\nu), m + m'), n),$$

and we must show that

$$(3.2) \quad \rho((\mu\alpha_{m'}(\nu), m + m'), n) = \rho((\mu, m'), n - m)\rho((\nu, m), n).$$

We calculate:

$$\begin{aligned}
\rho((\mu\alpha_{m'}(\nu), m + m'), n) &= (\alpha_{n-m-m'}(\mu\alpha_{m'}(\nu)), (n - m - m', n)) \\
&= (\alpha_{n-m-m'}(\mu)\alpha_{n-m}(\nu), (n - m - m', n - m)(n - m, n)) \\
&= (\alpha_{n-m-m'}(\mu), (n - m - m', n - m))(\alpha_{n-m}(\nu), (n - m, n)) \\
&= \rho((\mu, m'), n - m)\rho((\nu, m), n)
\end{aligned}$$

as required.  $\square$

One interpretation of Theorem 3.8 is that for an action of  $\mathbb{Z}^l$  on a  $k$ -graph  $C^*$ -algebra induced by an action of  $\mathbb{Z}^l$  on the  $k$ -graph itself, we may realise Takai Duality at the level of higher-rank graphs.

**Corollary 3.9.** *Let  $\alpha$  be an action of  $\mathbb{Z}^l$  on a row-finite  $k$ -graph  $\Lambda$  with no sources. Let  $\tilde{\alpha}$  denote the induced action of  $\mathbb{Z}^l$  on  $C^*(\Lambda)$ , and let  $\hat{\tilde{\alpha}}$  denote the dual action of  $\mathbb{T}^l$  on  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$ . The isomorphism  $\rho : (\Lambda \times_{\alpha} \mathbb{Z}^l) \times_c \mathbb{Z}^l \rightarrow \Lambda \times \Delta_l$  of Theorem 3.8 induces an isomorphism  $\tilde{\rho} : (C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l) \times_{\hat{\tilde{\alpha}}} \mathbb{T}^l \rightarrow C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\mathbb{Z}^l))$ .*

*Proof.* Corollary 3.5(iv) of [10] shows that  $C^*(\Lambda \times \Delta_l)$  is isomorphic to  $C^*(\Lambda) \otimes C^*(\Delta_l)$ , and as mentioned in Examples 3.7(3), the map  $s_{(m,n)} \mapsto \theta_{m,n}$  determines an isomorphism of  $C^*(\Delta_l)$  onto  $\mathcal{K}(\ell^2(\mathbb{Z}^l))$ .  $\square$

*Remark 3.10.* One can check that under appropriate conventions regarding dual actions and crossed products, the isomorphism obtained from Corollary 3.9 agrees with the Takai isomorphism as described in, for example, [23, Section 7.1].

#### 4. SIMPLICITY OF CROSSED PRODUCTS

In this section we investigate simplicity of  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  when  $\Lambda$  is row-finite and has no sources.

Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Recall from [10] that  $\Lambda$  is said to be *cofinal* if for every infinite path  $x \in \Lambda^{\infty}$  and every vertex  $v \in \Lambda^0$  there is a vertex  $x(n)$  on  $x$  such that  $v\Lambda x(n) \neq \emptyset$ . Recall from [21] that  $\Lambda$  has *no local periodicity* if for every vertex  $v \in \Lambda^0$  and each pair of distinct elements  $m, n \in \mathbb{N}^k$  there exists an infinite path  $x \in v\Lambda^{\infty}$  such that  $\sigma^m(x) \neq \sigma^n(x)$ . Lemma 3.3 of [21] implies that  $\Lambda$  has no local periodicity if and only if it satisfies the *aperiodicity condition* [10, Condition (A)].

Recall that if  $\phi \in \text{Aut}(\Lambda)$  is an automorphism of a  $k$ -graph  $\Lambda$ , then the formula

$$\phi^{\infty}(x)(0_k, m) := \phi(x(0_k, m)) \quad \text{for all } m \in \mathbb{N}^k$$

defines a range-preserving bijection of  $\Lambda^{\infty}$ .

**Definition 4.1.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and let  $\alpha$  be an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms.

- (1) We say that  $\Lambda$  is  $\alpha$ -*cofinal* if for every vertex  $v \in \Lambda^{0_k}$  and every infinite path  $x \in \Lambda^\infty$  there exist  $p \in \mathbb{N}^k$  and  $m, n \in \mathbb{N}^l$  such that  $\alpha_{-m}(v)\Lambda\alpha_{-n}(x(p)) \neq \emptyset$ .
- (2) We say that  $\Lambda$  is  $\alpha$ -*aperiodic* if, for each vertex  $v \in \Lambda^{0_k}$  and each pair of distinct elements  $(p, m)$  and  $(q, n)$  of  $\mathbb{N}^k \times \mathbb{N}^l$ , there is a path  $x \in v\Lambda^\infty$  such that  $\sigma^p(\alpha_{-m}^\infty(x)) \neq \sigma^q(\alpha_{-n}^\infty(x))$ .

**Theorem 4.2.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and let  $\alpha$  be an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. The crossed-product  $C^*$ -algebra  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^k$  is simple if and only if  $\Lambda$  is  $\alpha$ -cofinal and  $\alpha$ -aperiodic.*

To prove this theorem, we call upon the results of [21]. We begin by showing how the infinite paths of  $\Lambda$  correspond to those of  $\Lambda \times_\alpha \mathbb{Z}^l$ . Specifically, we show that each infinite path in  $\Lambda \times_\alpha \mathbb{Z}^l$  is determined by its restriction to  $\mathbb{N}^k$ .

**Lemma 4.3.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and suppose that  $\alpha$  is an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. Suppose that  $y, z \in (\Lambda \times_\alpha \mathbb{Z}^l)^\infty$  satisfy  $y((0_k, 0_l), (p, 0_l)) = z((0_k, 0_l), (p, 0_l))$  for all  $p \in \mathbb{N}^k$ . Then  $y = z$ .*

*Proof.* It suffices to show that  $y((0_k, 0_l), (p, m)) = z((0_k, 0_l), (p, m))$  for all  $(p, m) \in \mathbb{N}^k \times \mathbb{N}^l$ . To see this, fix  $(p, m) \in \mathbb{N}^k \times \mathbb{N}^l$ . Since  $y((0_k, 0_l), (p, 0_l))$  and  $z((0_k, 0_l), (p, 0_l))$  coincide by assumption, they have the same source. Since  $(v, 0_l)(\Lambda \times_\alpha \mathbb{Z}^l)^{(0_k, m)}$  is a singleton set for any fixed  $(v, 0_l) \in (\Lambda \times_\alpha \mathbb{Z}^l)^{0_{k+l}}$  and  $m \in \mathbb{N}^l$ , the paths  $y((p, 0_l), (p, m))$  and  $z((p, 0_l), (p, m))$  must also coincide. Hence

$$\begin{aligned} y((0_k, 0_l), (p, m)) &= y((0_k, 0_l), (p, 0_l))y((p, 0_l), (p, m)) \quad \text{and} \\ z((0_k, 0_l), (p, m)) &= (z(0_k, 0_l), (p, 0_l))z((p, 0_l), (p, m)) \end{aligned}$$

are identical as required.  $\square$

Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. As in [10, Section 2], the cylinder sets  $\lambda\Lambda^\infty$ ,  $\lambda \in \Lambda$  form a basis of compact open sets for a Hausdorff topology on  $\Lambda^\infty$ .

**Proposition 4.4.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and let  $\alpha$  be an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. Then there is a unique homeomorphism  $x \mapsto (x, \infty)$  from  $\Lambda^\infty$  onto  $(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$  such that*

$$(4.1) \quad (x, \infty)((0_k, 0_l), (p, 0_l)) = (x(0_k, p), 0_l) \quad \text{for all } p \in \mathbb{N}^k.$$

*Proof.* We first show that there exists a map  $x \mapsto (x, \infty)$  satisfying (4.1). To see this, fix  $x \in \Lambda^\infty$ , and define paths  $\{\lambda_{(p, m)} : (p, m) \in \mathbb{N}^{k+l}\} \subset \Lambda \times_\alpha \mathbb{Z}^l$  by  $\lambda_{(p, m)} := (x(0_k, p), m)$ . If  $(p, m) \leq (q, n) \in \mathbb{N}^{k+l}$ , then we have  $\lambda_{(q, n)}((0_k, 0_l), (p, m)) = \lambda_{(p, m)}$ , and it follows from [10, Remarks 2.2] that there is a unique infinite path  $(x, \infty) \in \Lambda \times_\alpha \mathbb{Z}^l$  such that  $(x, \infty)((0_k, 0_l), (p, m)) = \lambda_{(p, m)}$  for all  $(p, m) \in \mathbb{N}^{k+l}$ . Since  $\lambda_{(p, 0_l)}$  is precisely the right-hand side of (4.1), this establishes the existence of the desired map  $x \mapsto (x, \infty)$  from  $\Lambda^\infty$  to  $(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$ .

Lemma 4.3 guarantees that  $x \mapsto (x, \infty)$  is bijective and is the unique bijection satisfying (4.1).

It therefore remains only to show that  $x \mapsto (x, \infty)$  is a homeomorphism. To see this, observe that  $(\lambda, m)(\Lambda \times_\alpha \mathbb{Z}^l)^\infty = (\lambda, 0_l)(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$  for all  $\lambda \in \Lambda$  and  $m \in \mathbb{N}^l$ . In particular, the cylinder sets  $\{(\lambda, 0_l)(\Lambda \times_\alpha \mathbb{Z}^l)^\infty : \lambda \in \Lambda\}$  are a basis for the topology on  $(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$ , and since  $x \mapsto (x, \infty)$  restricts to a bijection of  $\lambda\Lambda^\infty$  onto  $(\lambda, 0_l)(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$ , it follows that  $x \mapsto (x, \infty)$  is a homeomorphism.  $\square$

The next lemma shows how to express the shift maps on  $(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$  in terms of the shift maps on  $\Lambda^\infty$  and the homeomorphisms  $\alpha_p^\infty$  of  $\Lambda^\infty$  obtained from the automorphisms  $\alpha_p$  of  $\Lambda$ .

**Lemma 4.5.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. and let  $\alpha$  be an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. Then for  $x \in \Lambda^\infty$  and  $(p, m) \in \mathbb{N}^k \times \mathbb{N}^l$ , the shift map on  $(\Lambda \times_\alpha \mathbb{Z}^l)^\infty$  satisfies  $\sigma^{(p, m)}(x, \infty) = (\alpha_{-m}^\infty(\sigma^p(x)), \infty)$ . Moreover,  $\sigma^p \circ \alpha_{-m}^\infty = \alpha_{-m}^\infty \circ \sigma^p$ ; in particular,  $\sigma^{(p, m)}(x, \infty) = (\sigma^p(\alpha_{-m}^\infty(x)), \infty)$ .*

*Proof.* Fix  $(p, m) \in \mathbb{N}^k \times \mathbb{N}^l$  and  $x \in \Lambda^\infty$ . Then  $\sigma^{(p, m)}(x, \infty) = \sigma^{(p, 0_l)}(\sigma^{(0_k, m)}(x, \infty))$ . For  $q \in \mathbb{N}^k$ , the initial segment of  $(x, \infty)$  of degree  $(q, m)$  is by definition equal to  $(r(x), m)(\alpha_{-m}(x(0_k, q), 0_l))$ . Hence  $\sigma^{(0_k, m)}(x, \infty) = (\alpha_{-m}^\infty(x), \infty)$ , and applying  $\sigma^{(p, 0_l)}$  to both sides, we obtain the desired identity  $\sigma^{(p, m)}(x, \infty) = (\alpha_{-m}^\infty(\sigma^p(x)), \infty)$ . To see that  $\alpha_{-m}^\infty \circ \sigma^p = \sigma^p \circ \alpha_{-m}^\infty$ , fix  $q \in \mathbb{N}^k$  and calculate:

$$(\alpha_{-m}^\infty(\sigma^p(x)))(0, q) = \alpha_{-m}(x(p, p+q)) = (\alpha_{-m}^\infty(x))(p, p+q) = (\sigma^p(\alpha_{-m}^\infty(x)))(0, q).$$

This completes the proof.  $\square$

**Lemma 4.6.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and  $\alpha$  an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. Then  $\Lambda$  is  $\alpha$ -cofinal if and only if  $\Lambda \times_\alpha \mathbb{Z}^l$  is cofinal*

*Proof.* For  $m \in \mathbb{N}^l$  and  $v, w \in \Lambda^{0_k}$ , we have  $\lambda \in \alpha_{-m}(v)\Lambda w$  if and only if  $(\alpha_m(\lambda), m) \in (v, 0_l)(\Lambda \times_\alpha \mathbb{Z}^l)(w, 0_l)$ . Hence for  $v, w \in \Lambda^{0_k}$ , we have

$$(4.2) \quad (v, 0_l)(\Lambda \times_\alpha \mathbb{Z}^l)(w, 0_l) \neq \emptyset \text{ if and only if } \alpha_{-m}(v)\Lambda w \neq \emptyset \text{ for some } m \in \mathbb{N}^l.$$

Moreover, for  $(p, n) \in \mathbb{N}^{k+l}$ , we have  $\alpha_{-n}(x(p)) = r(\alpha_{-n}^\infty(\sigma^p(x)))$ , so

$$(4.3) \quad (\alpha_{-n}(x(p)), 0_l) = \sigma^{p, n}(x, \infty)(0_k, 0_l) = (x, \infty)(p, n).$$

Recall that every vertex  $u$  of  $\Lambda \times_\alpha \mathbb{Z}^l$  is of the form  $(v, 0_l)$  for some  $v \in \Lambda^{0_k}$ . Proposition 4.4 shows that every infinite path  $y$  of  $\Lambda \times_\alpha \mathbb{Z}^l$  is of the form  $(x, \infty)$  for some  $x \in \Lambda^\infty$ . Thus (4.2) and (4.3) imply that  $\Lambda$  is  $\alpha$ -cofinal if and only if, for every vertex  $u \in (\Lambda \times_\alpha \mathbb{Z}^l)^{0_{k+l}}$  and every infinite path  $y \in (\Lambda \times_\alpha \mathbb{Z}^l)^\infty$ , there exists  $(p, n) \in \mathbb{N}^k \times \mathbb{N}^l$  such that  $u(\Lambda \times_\alpha \mathbb{Z}^l)y(p, n) \neq \emptyset$ , which is precisely the definition of cofinality of  $\Lambda \times_\alpha \mathbb{Z}^l$ .  $\square$

**Lemma 4.7.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and  $\alpha$  an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. Then  $\Lambda$  is  $\alpha$ -aperiodic if and only if  $\Lambda \times_\alpha \mathbb{Z}^l$  has no local periodicity in the sense of [21].*

*Proof.* The result follows from Lemma 4.5 and the definition of  $\alpha$ -aperiodicity.  $\square$



*Remark 4.8.* Suppose that the action  $\alpha$  is free in the sense that if  $\lambda \in \Lambda$  and  $n \in \mathbb{Z}^l$  satisfy  $\alpha_n(\lambda) = \lambda$ , then  $n = 0_l$ . As in [10, Section 5] we may form the quotient  $k$ -graph  $\Lambda/\mathbb{Z}^l$ , and [10, Theorem 5.7] shows that  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  is stably isomorphic to  $C^*(\Lambda/\alpha)$ . In particular, one can deduce from this, or from direct arguments, that  $\Lambda$  is  $\alpha$ -cofinal if and only if  $\Lambda/\alpha$  is cofinal, and  $\Lambda$  is aperiodic if and only if  $\Lambda/\alpha$  is aperiodic. In particular, when  $\alpha$  is free,  $\Lambda/\alpha$  is aperiodic (respectively cofinal) if and only if  $\Lambda \times_{\alpha} \mathbb{Z}^l$  is aperiodic (respectively cofinal).

If the action  $\alpha$  is not free then, as observed on [13, page 176], the natural definition of  $\Lambda/\alpha$  need not yield a category: the obvious candidate for a composition map is not necessarily well-defined. The approach of [10, Theorem 5.7] therefore cannot be applied to non-free actions. In particular, we cannot expect, in general, to be able to study  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  using a quotient  $k$ -graph. However, the crossed-product  $k$ -graph  $\Lambda \times_{\alpha} \mathbb{Z}^l$  makes sense regardless, and our results still apply.

*Proof of Theorem 4.2.* By Theorem 3.5, it suffices to show that  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  is simple if and only if  $\Lambda$  is both  $\alpha$ -aperiodic and  $\alpha$ -cofinal. By Lemmas 4.6 and 4.7, it therefore suffices to show that  $C^*(\Lambda \times_{\alpha} \mathbb{Z}^l)$  is simple if and only if  $\Lambda \times_{\alpha} \mathbb{Z}^l$  is cofinal and has no local periodicity. Since  $\Lambda \times_{\alpha} \mathbb{Z}^l$  is row-finite and has no sources, this follows from [21, Theorem 3.1].  $\square$

## 5. $C^*$ -ALGEBRAIC SIMPLICITY CRITERIA

In this section, we reinterpret the hypotheses that  $\Lambda$  is  $\alpha$ -aperiodic, and that  $\Lambda$  is  $\alpha$ -cofinal  $C^*$ -algebraically. Specifically, we re-cast these conditions in terms of the restriction of the induced action  $\tilde{\alpha}$  to the canonical abelian subalgebra  $\overline{D} = \overline{\text{span}}\{s_{\lambda}s_{\lambda}^* : \lambda \in \Lambda\}$  of  $C^*(\Lambda)$ . To do this, we insist that  $\Lambda$  should be *locally finite with no sources or sinks* in the sense that for each  $p \in \mathbb{N}^k$  and each  $v \in \Lambda^{0_k}$  there the sets  $v\Lambda^p$  and  $\Lambda^p v$  are both finite and nonempty. The resulting formulation is almost identical to [3, Proposition 8.29], and many of the ideas in the proof are drawn from that argument.

**Lemma 5.1.** *Let  $\Lambda$  be a locally finite  $k$ -graph with no sources or sinks. Let  $s_{\Lambda} = \{s_{\lambda} : \lambda \in \Lambda\} \subset C^*(\Lambda)$  denote the universal generating Cuntz-Krieger  $\Lambda$ -family. Let  $D$  be the  $*$ -subalgebra  $\text{span}\{s_{\lambda}s_{\lambda}^* : \lambda \in \Lambda\}$  of  $C^*(\Lambda)$ .*

*For  $a \in D$  and  $p \in \mathbb{N}^k$ , there are only finitely many paths  $\eta \in \Lambda^p$  such that  $s_{\eta}as_{\eta}^* \neq 0$ . Define  $\Phi_p : D \rightarrow D$  by*

$$(5.1) \quad \Phi_p(a) := \sum_{\eta \in \Lambda^p} s_{\eta}as_{\eta}^*.$$

*Then  $\|\Phi_p(a)\| = \|a\|$  for  $a \in D$ , and  $\Phi_p$  extends to an endomorphism, also denoted  $\Phi_p$  of  $\overline{D}$ . Moreover, the map  $p \mapsto \Phi_p$  defines an action of  $\mathbb{N}^k$  on  $\overline{D}$  by endomorphisms.*

*Proof.* We first show that  $\{\eta \in \Lambda^p : s_{\eta}as_{\eta}^* \neq 0\}$  is finite. First note that  $P_{r(G)} := \sum_{v \in r(G)} s_v$  is a left-identity for  $a$ . Relation (TCK1) ensures that  $s_w P_{r(G)} = 0$  for  $w \notin$

$r(G)$ . Relation (TCK2) therefore implies that  $s_\eta P_{r(G)} = 0$  whenever  $\eta \in \Lambda^p \setminus \Lambda^p r(G)$ . In particular,  $\{\eta \in \Lambda^p : s_\eta a s_\eta^* \neq 0\} \subset \Lambda^p r(G)$ . Since  $r(G)$  is finite, and since  $\Lambda$  is locally finite,  $\Lambda^p r(G)$  itself is finite. Thus  $\{\eta \in \Lambda^p : s_\eta a s_\eta^* \neq 0\}$  is finite as required.

It is clear that  $\Phi_p$  is linear and preserves adjoints. To see that  $\Phi_p$  extends to an endomorphism of  $\overline{D}$ , it suffices to show that  $\|\Phi_p(a)\| = \|a\|$  for each  $a \in D$ , and that  $\Phi_p(a)\Phi_p(b) = \Phi_p(ab)$  for  $a, b \in D$ .

Fix  $a \in D$ , and write  $a = \sum_{\lambda \in F} a_\lambda s_\lambda s_\lambda^* \in D$  where  $F \subset \Lambda$  is finite. Let  $q := \bigvee_{\lambda \in F} d(\lambda)$ . As  $\Lambda$  is row-finite with no sources, [19, Proposition B.1] implies that

$$(5.2) \quad s_v = \sum_{\lambda \in v\Lambda^q} s_\lambda s_\lambda^* \quad \text{for all } v \in \Lambda^0 \text{ and } q \in \mathbb{N}^k.$$

We may apply (5.2), to each term in  $a$  to obtain a finite set  $G \subset \Lambda^q$  and scalars  $\{b_\tau : \tau \in G\}$  such that  $a = \sum_{\tau \in G} b_\tau s_\tau s_\tau^*$ . Since the  $s_\tau s_\tau^*$  are mutually orthogonal projections, we have  $\|a\| = \max\{|b_\tau| : \tau \in G\}$ . By definition of  $\Phi$  and the Cuntz-Krieger relations we have

$$\Phi_p(a) = \sum_{\tau \in G, \eta \in \Lambda^p r(\tau)} b_\tau s_{\eta\tau} s_{\eta\tau}^*.$$

Since the  $s_{\eta\tau} s_{\eta\tau}^*$  are mutually orthogonal,  $\|\Phi_p(a)\| = \max\{|b_\tau| : \tau \in G, \Lambda^p r(\tau) \neq \emptyset\}$ . Since  $\Lambda$  has no sinks, this gives  $\|\Phi_p(a)\| = \|a\|$ .

To see that  $\Phi_p$  is multiplicative, fix  $a, b$  in  $D$  and use (5.2) to express  $a = \sum_{\tau \in F} a_\tau s_\tau s_\tau^*$  and  $b = \sum_{\rho \in G} b_\rho s_\rho s_\rho^*$  where  $F, G$  are finite subsets of  $\Lambda^q$  for some fixed  $q \in \mathbb{N}^k$ .

$$(5.3) \quad \begin{aligned} \Phi_p(a)\Phi_p(b) &= \sum_{\substack{\tau \in F, \eta \in \Lambda^p r(\tau) \\ \rho \in G, \zeta \in \Lambda^p r(\rho)}} s_\eta (a_\tau s_\tau s_\tau^*) s_\eta^* s_\zeta (b_\rho s_\rho s_\rho^*) s_\zeta^* \\ &= \sum_{\substack{\tau \in F, \eta \in \Lambda^p r(\tau) \\ \rho \in G, \zeta \in \Lambda^p r(\rho)}} s_\eta (a_\tau s_\tau s_\tau^* s_{\eta\tau}^* s_{\zeta\rho} b_\rho s_\rho^*) s_\zeta^*. \end{aligned}$$

Consider a product  $s_{\eta\tau}^* s_{\zeta\rho}$  occurring in a term in (5.3). Since  $F, G \subset \Lambda^q$ , (5.2) ensures that if the term  $s_{\eta\tau}^* s_{\zeta\rho}$  is nonzero, then  $\eta\tau = \zeta\rho$ . Since  $d(\eta) = d(\zeta) = p$ , the factorisation property guarantees that  $\eta\tau = \zeta\rho$  if and only if  $\eta = \zeta$  and  $\tau = \rho$ . Hence

$$(5.4) \quad \Phi_p(a)\Phi_p(b) = \sum_{\tau \in F \cap G, \eta \in \Lambda^p(r(\tau))} s_\eta (a_\tau b_\tau s_\tau^* s_\tau) s_\eta^*.$$

To see that (5.4) is equal to  $\Phi_p(ab)$ , we calculate

$$ab = \sum_{\tau \in F, \rho \in G} a_\tau b_\rho s_\tau s_\tau^* s_\rho s_\rho^* = \sum_{\tau \in F \cap G} a_\tau b_\tau s_\tau s_\tau^*$$

by (5.2). Applying the formula for  $\Phi_p$  to this expression, we obtain the right-hand side of (5.4).

It remains to show that  $p \mapsto \Phi_p$  determines an action of  $\mathbb{N}^k$ ; that is, we must show that  $\Phi_p \circ \Phi_q = \Phi_{p+q}$  for all  $p, q \in \mathbb{N}^k$ . By linearity, it suffices to show that  $\Phi_p(\Phi_q(s_\lambda s_\lambda^*)) = \Phi_{p+q}(s_\lambda s_\lambda^*)$  for all  $\lambda \in \Lambda$ . Fix  $\lambda \in \Lambda$ . Then

$$\Phi_p(\Phi_q(s_\lambda s_\lambda^*)) = \Phi_p\left(\sum_{\eta \in \Lambda^{qr}(\lambda)} s_{\eta\lambda} s_{\eta\lambda}^*\right) = \sum_{\eta \in \Lambda^{qr}(\lambda), \zeta \in \Lambda^{pr}(\eta)} s_{\zeta\eta\lambda} s_{\zeta\eta\lambda}^*.$$

The factorisation property implies that  $(\zeta, \eta) \mapsto \zeta\eta$  is a bijection of  $\{(\zeta, \eta) : \eta \in \Lambda^{qr}(\lambda), \zeta \in \Lambda^{pr}(\eta)\}$  onto  $\Lambda^{q+pr}(\lambda)$ . Hence

$$\Phi_p(\Phi_q(s_\lambda s_\lambda^*)) = \sum_{\xi \in \Lambda^{q+pr}(\lambda)} s_{\xi\lambda} s_{\xi\lambda}^* = \Phi_{q+p}(s_\lambda s_\lambda^*),$$

and  $\Phi$  is an action as claimed.  $\square$

**Proposition 5.2.** *Let  $\Lambda$  be a locally finite  $k$ -graph with no sources or sinks, and let  $\overline{D} = \overline{\text{span}}\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}$ . There is a unique isomorphism  $\psi : \overline{D} \rightarrow C_0(\Lambda^\infty)$  which takes  $s_\lambda s_\lambda^*$  to the indicator function  $1_{\lambda\Lambda^\infty}$ . For  $p \in \mathbb{N}^k$  and  $f \in C_0(\Lambda^\infty)$ , the endomorphism  $\Phi_p$  of  $\overline{D}$  obtained from Lemma 5.1 satisfies  $\psi \circ \Phi_p \circ \psi^{-1}(f) = f \circ \sigma^p$  as elements of  $C_0(\Lambda^\infty)$ . If  $\alpha$  is an action of  $\mathbb{Z}^l$  by automorphisms of  $\Lambda$ , then for  $m \in \mathbb{Z}^l$  and  $f \in C_0(\Lambda^\infty)$ , the automorphism  $\tilde{\alpha}_m$  of  $\overline{D}$  satisfies  $\psi \circ \tilde{\alpha}_m \circ \psi^{-1}(f) = f \circ \alpha_{-m}^\infty$ .*

*Proof.* The existence of the isomorphism  $\psi$  follows from [10, Corollary 3.5(i)];  $\psi$  is unique because  $\overline{D}$  is generated as a  $C^*$ -algebra by the  $s_\lambda s_\lambda^*$ . For the last assertion it suffices to show that for  $\lambda \in \Lambda$

$$\psi \circ \Phi_p \circ \psi^{-1}(1_{\lambda\Lambda^\infty})(x) = 1_{\lambda\Lambda^\infty} \circ \sigma^p(x) \quad \text{for } x \in \Lambda^\infty,$$

and similarly for  $\tilde{\alpha}_m$  and  $\alpha_m^\infty$ . Fix  $x \in \Lambda^\infty$ . Since

$$\begin{aligned} \psi \circ \Phi_p \circ \psi^{-1}(1_{\lambda\Lambda^\infty}) &= \psi(\Phi_p(s_\lambda s_\lambda^*)) \\ &= \psi\left(\sum_{\xi \in r(\lambda)\Lambda^p} s_{\xi\lambda} s_{\xi\lambda}^*\right) \\ &= \sum_{\xi \in r(\lambda)\Lambda^p} 1_{\xi\lambda}, \end{aligned}$$

we may calculate

$$\begin{aligned} \psi \circ \Phi_p \circ \psi^{-1}(1_{\lambda\Lambda^\infty})(x) &= \left(\sum_{\xi \in r(\lambda)\Lambda^p} 1_{\xi\lambda}\right)(x) \\ &= \begin{cases} 1 & \text{if } x(0_k, p + d(\lambda)) = \xi\lambda \text{ for some } \xi \in r(\lambda)\Lambda^p \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \sigma^p(x)(0_k, d(\lambda)) = \lambda \\ 0 & \text{otherwise} \end{cases} \\ &= 1_{\lambda\Lambda^\infty}(\sigma^p(x)). \end{aligned}$$

A similar argument establishes the identity involving  $\tilde{\alpha}_m$  and  $\alpha_{-m}^\infty$ .  $\square$

Suppose that  $\Lambda$  is a locally finite  $k$ -graph with no sources or sinks, and that  $\alpha$  is an action of  $\mathbb{Z}^l$  on  $\Lambda$  by automorphisms. For  $(p, m) \in \mathbb{N}^k \times \mathbb{N}^l$ , the map  $\sigma^p \circ \alpha_{-m}^\infty$  is a local homeomorphism of  $\Lambda^\infty$ . We denote this local homeomorphism by  $\tau_{p,m}^{\sigma,\alpha}$ . Then  $\tau^{\sigma,\alpha} : (p, m) \mapsto \tau_{p,m}^{\sigma,\alpha}$  is an action of the semigroup  $\mathbb{N}^k \times \mathbb{N}^l$  by local homeomorphisms of  $\Lambda^\infty$ .

Let  $X$  be a topological space, and let  $\tau$  be an action of a semigroup  $S$  by local homeomorphisms of  $X$ . As in [1, 3],

- (1) We say that the system  $(X, \tau)$  is *topologically free* if for every pair of distinct elements  $s, t \in S$ , the set  $\{x \in X : \tau_s(x) = \tau_t(x)\}$  has empty interior.
- (2) We say that  $x, y \in X$  are *trajectory equivalent* if there exist  $s, t \in S$  such that  $\tau_s(x) = \tau_t(y)$ .
- (3) We say that  $W \subset X$  is *invariant* if  $y \in W$  and  $x$  trajectory equivalent to  $y$  imply  $x \in W$ .
- (4) We say that  $\tau$  is *irreducible* if the only open invariant subsets of  $X$  are  $\emptyset$  and  $X$ .

*Remark 5.3.* Suppose that  $S = \mathbb{N}^k \times \mathbb{N}^l$ . Then trajectory equivalence is an equivalence relation: it is clearly reflexive and symmetric, and to see that it is transitive, suppose that  $x, y, z \in X$ ,  $p, q, p', q' \in \mathbb{N}^k$  and  $m, n, m', n' \in \mathbb{N}^l$  satisfy  $\tau_{(p,m)}(x) = \tau_{(q,n)}(y)$  and  $\tau_{(p',m')}(y) = \tau_{(q',n')}(z)$ . Then

$$\begin{aligned} \tau_{(p+(q \vee q')-q, m+(n \vee n')-n)}(x) &= \tau_{(q \vee q')-q, (n \vee n')-n}(\tau_{(p,m)}(x)) \\ &= \tau_{(q \vee q')-q, (n \vee n')-n}(\tau_{(q,n)}(y)) \\ &= \tau_{(q,m) \vee (q',n')}(y). \end{aligned}$$

Symmetrically,  $\tau_{(p+(q \vee q')-q', m+(n \vee n')-n')}(z) = \tau_{(q,m) \vee (q',n')}(y)$ , so that

$$\tau_{(p+(q \vee q')-q, m+(n \vee n')-n)}(x) = \tau_{(p+(q \vee q')-q', m+(n \vee n')-n')}(z).$$

In particular, a set  $U \subset X$  is invariant if and only if its complement  $X \setminus U \subset X$  is invariant, so  $\tau$  is irreducible if and only if the only closed invariant subsets of  $X$  are  $\emptyset$  and  $X$ .

**Theorem 5.4.** *Let  $\Lambda$  be a locally finite  $k$ -graph with no sources or sinks, and let  $\alpha$  be an action of  $\mathbb{Z}^l$  by automorphisms of  $\Lambda$ . Then*

- (1) *Every nontrivial ideal  $I$  of  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  satisfies  $I \cap \pi_{C^*(\Lambda)}(\overline{D}) \neq \{0\}$  if and only if  $(\Lambda^\infty, \tau^{\sigma,\alpha})$  is topologically free.*
- (2) *The ideals  $I(a)$  in  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  generated by nonzero elements  $a$  of  $\pi_{C^*(\Lambda)}(\overline{D})$  are all equal to  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  if and only if  $\tau^{\sigma,\alpha}$  is irreducible.*

*In particular  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  is simple if and only if  $(\Lambda^\infty, \tau^{\sigma,\alpha})$  is topologically free and  $\tau^{\sigma,\alpha}$  is irreducible.*

*Remark 5.5.* The above theorem applies when  $l = 0$  so that  $\alpha$  is the trivial action of the trivial group  $\{0\}$ . In this case,  $\tau^{\sigma,\alpha}$  is just the action  $\sigma$  of  $\mathbb{N}^k$  on  $\Lambda^\infty$  by shift maps, and we obtain a parallel result to [3, Proposition 8.29] for locally-finite  $k$ -graphs.

To prove Theorem 5.4 we establish two lemmas. The first establishes that topological freeness is equivalent to  $\alpha$ -aperiodicity, and the second that irreducibility is equivalent to  $\alpha$ -cofinality. We then apply Theorem 4.2 to obtain the result.

**Lemma 5.6.** *Let  $\Lambda$  be a locally finite  $k$ -graph with no sources or sinks, and let  $\alpha$  be an action of  $\mathbb{Z}^l$  by automorphisms of  $\Lambda$ . Then  $\Lambda$  is  $\alpha$ -aperiodic if and only if  $(\Lambda^\infty, \tau^{\sigma, \alpha})$  is topologically free.*

*Proof.* First suppose that  $(\Lambda^\infty, \tau^{\sigma, \alpha})$  is topologically free. Fix  $v \in \Lambda^{0_k}$  and  $(p, m) \neq (q, n) \in \mathbb{N}^k \times \mathbb{N}^l$ . Since  $v\Lambda^\infty$  is open in  $\Lambda^\infty$ , topological freeness ensures that there exists an  $x \in v\Lambda^\infty$  such that  $\tau_{p,m}^{\sigma, \alpha}(x) \neq \tau_{q,n}^{\sigma, \alpha}(x)$ ; that is  $\sigma^p(\alpha_{-m}^\infty(x)) \neq \sigma^q(\alpha_{-n}^\infty(x))$ . Since  $v, (p, m)$  and  $(q, n)$  were arbitrary, it follows that  $\Lambda$  is  $\alpha$ -aperiodic.

Now suppose that  $\Lambda$  is  $\alpha$ -aperiodic. We must show that  $(\Lambda^\infty, \tau^{\sigma, \alpha})$  is topologically free. Fix  $(p, m) \neq (q, n) \in \mathbb{N}^k \times \mathbb{N}^l$ . We must show that  $\{x \in \Lambda^\infty : \sigma^p(\alpha_{-m}^\infty(x)) = \sigma^q(\alpha_{-n}^\infty(x))\}$  has empty interior. Since the sets  $\lambda\Lambda^\infty, \lambda \in \Lambda$  form a basis for the topology on  $\Lambda^\infty$ , it suffices to show that for each fixed  $\lambda\Lambda^\infty$ , there exists  $x \in \lambda\Lambda^\infty$  such that  $\sigma^p(\alpha_{-m}^\infty(x)) \neq \sigma^q(\alpha_{-n}^\infty(x))$ . Fix  $\lambda \in \Lambda$ . By  $\alpha$ -aperiodicity, there exists  $y \in s(\lambda)\Lambda^\infty$  such that  $\sigma^p(\alpha_{-m}^\infty(y)) \neq \sigma^q(\alpha_{-n}^\infty(y))$ , and then  $x := \lambda y$  has the desired property.  $\square$

**Lemma 5.7.** *Let  $\Lambda$  be a locally finite  $k$ -graph with no sources or sinks, and let  $\alpha$  be an action of  $\mathbb{Z}^l$  by automorphisms of  $\Lambda$ . Then  $\Lambda$  is  $\alpha$ -cofinal if and only if  $\tau^{\sigma, \alpha}$  is irreducible.*

*Proof.* We follow the proof of [3, Lemma 8.31] quite closely. First suppose that  $\Lambda$  is  $\alpha$ -cofinal. Let  $U$  be a nonempty open invariant subset of  $\Lambda^\infty$ ; we must show that  $U = \Lambda^\infty$ . Since  $U$  is open and nonempty, there exists  $\lambda \in \Lambda$  such that  $\lambda\Lambda^\infty \subset U$ . Fix  $x \in \Lambda^\infty$ . Since  $\Lambda$  is  $\alpha$ -cofinal, there exist  $p \in \mathbb{N}^k, m, n \in \mathbb{N}^l$  and  $\mu \in \Lambda$  such that  $r(\mu) = \alpha_{-m}(s(\lambda))$  and  $s(\mu) = \alpha_{-n}(x(p))$ . We have  $y := \lambda\alpha_m(\mu)\alpha_{m-n}^\infty(\sigma^p(x)) \in \lambda\Lambda^\infty \subset U$ . Moreover,

$$\tau_{d(\lambda)+d(\mu), n}^{\sigma, \alpha}(y) = \alpha_n^\infty(\alpha_{m-n}^\infty(\sigma^p(x))) = \tau_{p,m}^{\sigma, \alpha}(x),$$

so that  $x$  and  $y$  are trajectory equivalent. Since  $U$  is invariant, this forces  $x \in U$ . Since  $x \in \Lambda^\infty$  was arbitrary, it follows that  $U = \Lambda^\infty$ .

Now suppose that  $\Lambda$  is not  $\alpha$ -cofinal, and fix  $v \in \Lambda^{0_k}$  and  $x \in \Lambda^\infty$  such that  $\alpha_{-m}(v)\Lambda\alpha_{-n}(x(p)) = \emptyset$  for all  $p \in \mathbb{N}^k$  and  $m, n \in \mathbb{N}^l$ . We will show that  $\tau^{\sigma, \alpha}$  is not irreducible by constructing an open invariant set  $U$  which is equal to neither  $\Lambda^\infty$  nor  $\emptyset$ . Let

$$U := \{y \in \Lambda^\infty : \alpha_{-m}(v)\Lambda\alpha_{-n}(y(p)) \neq \emptyset \text{ for some } p \in \mathbb{N}^k \text{ and } m, n \in \mathbb{N}^l\}.$$

Since  $v\Lambda^\infty \neq \emptyset$ , we have  $U \neq \emptyset$ , and since  $x \notin U$  by construction, we have  $U \neq \Lambda^\infty$ .

We claim that  $U$  is open. To see this, fix  $z \in U$ , and let  $p \in \mathbb{N}^k$  and  $m, n \in \mathbb{N}^l$  satisfy  $\alpha_{-m}(v)\Lambda\alpha_{-n}(z(p)) \neq \emptyset$ . Let  $\lambda := z(0_k, p)$ . For any  $z' \in \lambda\Lambda^\infty$  we have  $\alpha_{-n}(z'(p)) = \alpha_{-n}(s(\lambda)) = \alpha_{-n}(z(p))$ , and hence  $\lambda\Lambda^\infty \subset U$ . Since  $\lambda\Lambda^\infty$  is an open neighbourhood of  $z$ , it follows that  $U$  is open.

Finally, we claim that  $U$  is invariant. Suppose that  $y \in U$  and that  $z$  is trajectory equivalent to  $y$ . Fix  $p \in \mathbb{N}^k$  and  $m, n \in \mathbb{N}^l$  such that  $\alpha_{-m}(v)\Lambda\alpha_{-n}(y(p)) \neq \emptyset$ . Then

$$(5.5) \quad \alpha_{-(m+h)}(v)\Lambda\alpha_{-(n+h)}(y(p+l)) \neq \emptyset \text{ for all } l \in \mathbb{N}^k, h \in \mathbb{N}^l.$$

Since  $y$  is trajectory equivalent to  $z$ , there exist  $c, d \in \mathbb{N}^k$  and  $a, b \in \mathbb{N}^l$  such that  $\tau_{c,a}^{\sigma,\alpha}(y) = \tau_{d,b}^{\sigma,\alpha}(z)$ ; that is,  $\alpha_{-a}(y(c)) = \alpha_{-b}(z(d))$ , and we may assume without loss of generality that  $a \geq n$  and  $c \geq p$ ; say  $a = n + h$  and  $c = p + l$ . By (5.5), we then have

$$\emptyset \neq \alpha_{-(m+h)}(v)\Lambda\alpha_{-(n+h)}(y(p+l)) = \alpha_{-(m+h)}(v)\Lambda\alpha_{-a}(y(c)),$$

and as  $\alpha_{-a}(y(c)) = \alpha_{-b}(z(d))$  by choice of  $a, b, c, d$ , it follows that  $z \in U$ . Hence  $U$  is invariant, and the proof is complete.  $\square$

*Proof of Theorem 5.4.* The last assertion follows from (1) and (2). By definition of  $\overline{D}$ , every ideal of  $C^*(\Lambda)$  which intersects  $\overline{D}$  must contain a vertex projection. Hence it suffices to show: (a) that every ideal of  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  contains a vertex projection if and only if  $(\Lambda^\infty, \tau^{\sigma,\alpha})$  is topologically free; and (b) that the ideals of  $C^*(\Lambda) \times_{\tilde{\alpha}} \mathbb{Z}^l$  generated by the projections  $\pi_{C^*(\Lambda)}(s_v)$  are all equal to the whole crossed product if and only if  $\tau^{\sigma,\alpha}$  is irreducible. But (a) follows from Lemmas 5.6 and 4.7 together with [21, Proposition 3.6] and (b) follows from Lemmas 5.7 and 4.6 together with [21, Proposition 3.5].  $\square$

## 6. $K$ -THEORY

In this section we consider an action of  $\mathbb{Z}$  on a row-finite 1-graph  $E$  with no sources such that either  $K_0(C^*(E)) = \{0\}$  or  $K_1(C^*(E)) = \{0\}$ . In this case,  $C^*(E \times_\alpha \mathbb{Z})$  is a crossed product of  $C^*(E)$  by  $\mathbb{Z}$  and we can use the Pimsner-Voiculescu exact sequence to investigate its  $K$ -theory.

Our main application is to the 2-graphs discussed in [15, Section 3]. These can be realised, using Proposition 3.6, as 2-graphs of the form  $E \times_\alpha \mathbb{Z}$  where  $E$  is a 1-graph with no cycles. This guarantees that  $C^*(E)$  is an AF algebra, so has trivial  $K_1$ -group, and the Pimsner-Voiculescu sequence provides a relatively straightforward calculation of the  $K$ -theory of the crossed-product algebra  $C^*(E) \times_{\tilde{\alpha}} \mathbb{Z}$ . This approach is significantly more efficient than the calculations of [15, Section 4], even for the smaller class of rank-2 Bratteli diagrams considered there.

To state the main result, we need some notation and definitions.

Given a set  $X$ , we let  $\mathbb{Z}X$  denote the collection of finitely supported functions  $f : X \rightarrow \mathbb{Z}$ , and regard it as a group under pointwise addition. We write  $\{\delta_x : x \in X\}$  for the canonical basis for  $\mathbb{Z}X$ .

Let  $E$  be a row-finite 1-graph with no sources. Let  $M_E$  denote the connectivity matrix of  $E$  given by  $M_E(v, w) = |\{e \in E^1 : r(e) = v, s(e) = w\}|$ . We regard  $M_E$  as a homomorphism of  $\mathbb{Z}E^0$  (implemented by matrix multiplication).

Given an automorphism  $\alpha$  of  $E$ , we write  $\alpha_*$  for the induced homomorphism  $\alpha_* : \mathbb{Z}E^0 \rightarrow \mathbb{Z}E^0$  determined by  $\alpha_*(f)(v) = f(\alpha(v))$ . Equivalently,  $\alpha_*(\delta_v) = \delta_{\alpha^{-1}(v)}$  for all  $v \in E^0$ .

**Theorem 6.1.** *Let  $E$  be a row-finite 1-graph with no sources, and let  $\alpha$  be an automorphism of  $E$ . Resume the notation outlined above. Then  $\alpha_*$  commutes with  $M_E^t$  and induces an automorphism  $\widetilde{\alpha}_*$  of  $\text{coker}(1 - M_E^t)$  satisfying*

$$\widetilde{\alpha}_*(f + \text{im}(1 - M_E^t)) := \alpha_*(f) + \text{im}(1 - M_E^t).$$

Furthermore,  $\alpha_*$  restricts to an automorphism  $\alpha_*|$  of  $\ker(1 - M_E^t)$ .

There is an isomorphism  $\phi_0 : K_0(C^*(E)) \rightarrow \text{coker}(1 - M_E^t)$  which satisfies  $\phi_0([s_v]) = \delta_v + \text{im}(1 - M_E^t)$  and there is an isomorphism  $\phi_1 : K_1(C^*(E)) \rightarrow \ker(1 - M_E^t)$  such that the diagrams

$$\begin{array}{ccc} K_0(C^*(E)) & \xrightarrow{\phi_0} & \text{coker}(1 - M_E^t) \\ \downarrow K_0(\tilde{\alpha}) & & \downarrow \widetilde{\alpha}_* \\ K_0(C^*(E)) & \xrightarrow{\phi_0} & \text{coker}(1 - M_E^t) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_1(C^*(E)) & \xrightarrow{\phi_1} & \ker(1 - M_E^t) \\ \downarrow K_1(\tilde{\alpha}) & & \downarrow \alpha_*| \\ K_1(C^*(E)) & \xrightarrow{\phi_1} & \ker(1 - M_E^t) \end{array}$$

commute.

*Proof.* To see that  $\alpha_*$  commutes with  $M_E^t$ , fix a generator  $\delta_v$  of  $\mathbb{Z}E^0$  calculate:

$$\begin{aligned} \alpha_*(M_E^t \delta_v) &= \alpha_*\left(\sum_{r(e)=v} \delta_{s(e)}\right) = \sum_{r(e)=v} \delta_{\alpha^{-1}(s(e))} \\ &= \sum_{r(e')=\alpha^{-1}(v)} \delta_{s(e')} = M_E^t \delta_{\alpha^{-1}(v)} = M_E^t(\alpha_*(\delta_v)). \end{aligned}$$

The remaining statements follow from the definitions of the maps  $\tilde{\alpha}$  and  $\widetilde{\alpha}_*$ , the  $K$ -theory calculations for graph algebras of [14, 20], and the naturality of the Pimsner-Voiculescu exact sequence (see for example [20, Section 3]).  $\square$

**Corollary 6.2.** *Resume the notation of Theorem 6.1.*

- (1) *Suppose that  $K_1(C^*(E)) = \{0\}$ . Then  $K_0(C^*(E \times_\alpha \mathbb{Z})) \cong \text{coker}(1 - \widetilde{\alpha}_*)$  and  $K_1(C^*(E \times_\alpha \mathbb{Z})) \cong \ker(1 - \widetilde{\alpha}_*)$ .*
- (2) *Suppose that  $K_0(C^*(E)) = \{0\}$ . Then  $K_0(C^*(E \times_\alpha \mathbb{Z})) \cong \ker(1 - \alpha_*|_{\ker(1 - M_E^t)})$  and  $K_1(C^*(E \times_\alpha \mathbb{Z})) \cong \text{coker}(1 - \alpha_*|_{\ker(1 - M_E^t)})$ .*

*Proof.* The result follows immediately from the Pimsner-Voiculescu exact sequence for the action  $\tilde{\alpha}$  (see [16, Theorem 2.4]).  $\square$

**Notation 6.3.** Let  $E$  be a row-finite 1-graph with no sources, and let  $\alpha$  be an action of  $\mathbb{Z}$  on  $E$  by automorphisms such that the orbit of each vertex under  $\alpha$  is finite. For each  $v \in E^0$ , let  $C(v) := \{\alpha_n(v) : n \in \mathbb{Z}\}$  be the orbit of  $v$ . Let  $\mathcal{C} := \{C(v) : v \in E^0\}$  be the collection of all orbits of vertices under  $\alpha$ . Define integer-valued  $\mathcal{C} \times \mathcal{C}$  matrices  $A$  and  $B$  as follows: for  $C_1, C_2 \in \mathcal{C}$ , define

$$A_{C_1, C_2} := |C_1 E^1 C_2| / |C_1| \quad \text{and} \quad B_{C_1, C_2} := |C_1 E^1 C_2| / |C_2|.$$

An argument like the proof of [15, Lemma 4.2], shows that for any  $v_1 \in C_1$  and  $v_2 \in C_2$ , we have  $A_{C_1, C_2} = |v_1 E^1 C_2|$  and  $B_{C_1, C_2} = |C_1 E^1 v_2|$ . We regard  $A$  and  $B$  as homomorphisms of  $\mathbb{Z}\mathcal{C}$  regarded as a group under addition.

**Proposition 6.4.** *Let  $E$  be a row-finite 1-graph with no sources, and let  $\alpha$  be an action of  $\mathbb{Z}$  on  $E$  by automorphisms. Suppose that the orbit of each vertex  $v \in E^0$  under  $\alpha$  is finite, and let  $\mathcal{C}$ , and  $A, B : \mathbb{Z}\mathcal{C} \rightarrow \mathbb{Z}\mathcal{C}$  be as in Notation 6.3.*

(1) *If  $K_1(C^*(E)) = \{0\}$ , then*

$$K_0(C^*(E \times_\alpha \mathbb{Z})) \cong \text{coker}(1 - A^t) \quad \text{and} \quad K_1(C^*(E \times_\alpha \mathbb{Z})) \cong \text{coker}(1 - B^t).$$

(2) *If  $K_0(C^*(E)) = \{0\}$ , then*

$$K_0(C^*(E \times_\alpha \mathbb{Z})) \cong \ker(1 - B^t) \quad \text{and} \quad K_1(C^*(E \times_\alpha \mathbb{Z})) \cong \ker(1 - A^t).$$

*Proof.* Let  $M_E$  be the adjacency matrix of  $E$ . Let  $\phi := 1 - M_E^t : \mathbb{Z}E^0 \rightarrow \mathbb{Z}E^0$ , and let  $\psi := 1 - \alpha_* : \mathbb{Z}E^0 \rightarrow \mathbb{Z}E^0$ .

The strategy is to define maps  $\phi|$ ,  $\tilde{\phi}$ ,  $\tilde{\psi}$ ,  $q_\phi$ ,  $q_\psi$ ,  $q_{\phi|}$ ,  $q_{\tilde{\phi}}$ ,  $q_{\tilde{\psi}}$  and  $\tilde{q}_\psi$  which make the 16-term diagram illustrated in Figure 1 commute. Under the hypothesis that one of

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \ker(\phi) \cap \ker(\psi) & \hookrightarrow & \ker(\phi) & \xrightarrow{\psi|} & \ker(\phi) & \xrightarrow{q_\psi|} & \ker(\tilde{\phi}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \ker(\psi) & \hookrightarrow & \mathbb{Z}E^0 & \xrightarrow{\psi} & \mathbb{Z}E^0 & \xrightarrow{q_\psi} & \text{coker}(\psi) & \longrightarrow & 0 \\
 & & \downarrow \phi| & & \downarrow \phi & & \downarrow \phi & & \downarrow \tilde{\phi} & & \\
 0 & \longrightarrow & \ker(\psi) & \hookrightarrow & \mathbb{Z}E^0 & \xrightarrow{\psi} & \mathbb{Z}E^0 & \xrightarrow{q_\psi} & \text{coker}(\psi) & \longrightarrow & 0 \\
 & & \downarrow q_{\phi|} & & \downarrow q_\phi & & \downarrow q_\phi & & \downarrow q_{\tilde{\phi}} & & \\
 0 & \longrightarrow & \text{coker}(\phi|) & \xrightarrow{\pi} & \text{coker}(\phi) & \xrightarrow{\tilde{\psi}} & \text{coker}(\phi) & \xrightarrow{\tilde{q}_\psi} & \text{coker}(\tilde{\phi}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 
 \end{array}$$

FIGURE 1. The 16-term diagram for  $\phi$  and  $\psi$ .

$K_0(C^*(E))$ ,  $K_1(C^*(E))$  is trivial, we will use the Sixteen Lemma to deduce that all rows and columns of Figure 1 are exact and hence that

$$(6.1) \quad \ker(1 - \widetilde{\alpha_*}) \cong \text{coker}(\phi|), \quad \text{and} \quad \text{coker}(1 - \widetilde{\alpha_*}) \cong \text{coker}(\tilde{\phi}),$$

$$(6.2) \quad \ker((1 - \alpha_*)|_{\ker(\phi)}) \cong \ker(\phi|), \quad \text{and} \quad \text{coker}((1 - \alpha_*)|_{\ker(\phi)}) \cong \ker(\tilde{\phi}).$$

Finally, we will establish the existence of isomorphisms  $\theta_A : \text{coker}(\psi) \rightarrow \mathbb{Z}\mathcal{C}$  and  $\theta_B : \ker(\psi) \rightarrow \mathbb{Z}\mathcal{C}$  satisfying

$$\theta_A \circ \tilde{\phi} = (1 - A^t)\theta_A \quad \text{and} \quad \theta_B \circ \phi| = (1 - B^t)\theta_B.$$



Combining these with Corollary 6.2 will complete the proof.

To define the maps  $\phi|$ ,  $\tilde{\phi}$  and  $\tilde{\psi}$  in Figure 1, recall that Theorem 6.1 implies that  $M_E^t$  and  $\alpha_*$  commute, and hence that  $\phi$  and  $\psi$  commute. It follows that  $\phi$  restricts to a homomorphism  $\phi|$  of the kernel of  $\psi$  and induces a homomorphism  $\tilde{\phi}$  of  $\text{coker}(\psi)$  which satisfies

$$\tilde{\phi}(f + \psi(\mathbb{Z}E^0)) = \phi(f) + \psi(\mathbb{Z}E^0).$$

In a similar fashion,  $\psi$  restricts to a homomorphism  $\psi|$  of  $\ker(\phi)$  and induces a homomorphism  $\tilde{\psi}$  of  $\text{coker}(\phi)$ .

The maps  $q_\phi$ ,  $q_\psi$ ,  $q_{\phi|}$  and  $q_{\tilde{\phi}}$  in Figure 1 are the natural quotient maps: for example,  $q_\phi$  is defined by  $q_\phi(f) = f + \phi(\mathbb{Z}E^0)$ .

The homomorphism  $\pi$  in Figure 1 is defined by  $\pi(f + \phi(\ker(\psi))) := f + \phi(\mathbb{Z}E^0)$  for  $f \in \mathbb{Z}E^0$ ; this is well-defined because  $\phi(\ker(\psi)) \subset \phi(\mathbb{Z}E^0)$ .

The map  $q_{\psi|}$  in Figure 1 is the restriction of  $q_\psi$  to the kernel of  $\phi$ ; this has range in  $\ker(\tilde{\phi})$  by definition of  $\tilde{\phi}$ .

The map  $\tilde{q}_\psi$  in Figure 1 is defined by  $\tilde{q}_\psi(f + \phi(\mathbb{Z}E^0)) := q_\psi(f) + \tilde{\phi}(\text{coker}(\psi))$ . To see that this is well-defined, note that for  $f \in \phi(\mathbb{Z}E^0)$ , we have  $f = \phi(g)$  for some  $g \in \mathbb{Z}E^0$ . Since  $q_\psi(f) = q_\psi(\phi(g)) = \tilde{\phi}(\psi(g))$  by definition of  $q_{\tilde{\phi}}$ , it follows that  $q_\psi(f) \in \tilde{\phi}(\text{coker}(\psi))$  as required.

We now need to show that the squares in Figure 1 commute, and that if one of  $K_0(C^*(E))$  or  $K_1(C^*(E))$  is trivial, then all rows and columns are exact.

As mentioned above, Theorem 6.1 shows that  $\alpha_*$  and  $M_E^t$  commute, and it follows that the middle square of Figure 1 commutes. The other squares commute by definition of the maps involved.

The middle two rows and all columns are clearly exact. Theorem 4.2.4 of [14] shows that  $\ker(\phi) \cong K_1(C^*(E))$ , and  $\text{coker}(\phi) \cong K_0(C^*(E))$ . Hence if  $K_1(C^*(E)) = \{0\}$ , then the terms in the top row are all equal to  $\{0\}$ , and that row is trivially exact, and likewise if  $K_0(C^*(E)) = \{0\}$ , then the terms in the bottom row are all equal to  $\{0\}$  and that row is trivially exact.

In either case, we may apply the Sixteen Lemma to deduce that the remaining row of the diagram is exact. The exactness of the top and bottom rows establishes the formulae (6.1) and (6.2).

By Corollary 6.2, it therefore suffices to show that  $\text{coker}(\tilde{\phi}) \cong \text{coker}(1 - A^t)$ ,  $\text{coker}(\phi|) \cong \text{coker}(1 - B^t)$ ,  $\ker(\tilde{\phi}) \cong \ker(1 - A^t)$  and  $\ker(\phi|) \cong \ker(1 - B^t)$ .

Let  $V$  be a subset of  $E^0$  which contains exactly one representative of each orbit  $C \in \mathcal{C}$ . Then  $\text{coker}(\psi)$  is generated by the classes  $\{\delta_v + \psi(\mathbb{Z}E^0) : v \in V\}$ , and  $\delta_v - \delta_w \in \psi(\mathbb{Z}E^0)$  if and only if  $C(v) = C(w)$ . Hence there is an isomorphism  $\theta_A : \text{coker}(\psi) \rightarrow \mathbb{Z}\mathcal{C}$  such that  $\theta_A(\delta_v + \psi(\mathbb{Z}E^0)) = \delta_{C(v)}$  for all  $v \in E^0$ .

Now  $\tilde{\phi}$  takes  $\delta_v + \psi(\mathbb{Z}E^0)$  to  $\sum_{r(e)=v} \delta_{s(e)} + \psi(\mathbb{Z}E^0)$ . Applying  $\theta_A$ , we have

$$\begin{aligned} \theta_A(\tilde{\phi}(\delta_v + \psi(\mathbb{Z}E^0))) &= \sum_{r(e)=v} \delta_{C(s(e))} \\ (6.3) \qquad \qquad \qquad &= \sum_{C \in \mathcal{C}} |\{e \in r^{-1}(v) : C = C(s(e))\}| \delta_C. \end{aligned}$$

The expression (6.3) is equal to  $(1-A^t)(\delta_{C(v)})$  by definition of  $A$ , and  $(1-A^t)(\delta_{C(v)}) = (1-A^t)(\theta_A(\delta_v + \psi(\mathbb{Z}E^0)))$  by definition of  $\theta_A$ . Hence the isomorphism  $\theta_A$  intertwines  $(1-A^t)$  and  $\tilde{\phi}$ , establishing that  $\text{coker}(\tilde{\phi}) \cong \text{coker}(1-A^t)$  and  $\ker(\tilde{\phi}) \cong \ker(1-A^t)$ .

Next note that the map  $\alpha_*$  permutes the point-masses associated to the vertices in each  $C \in \mathcal{C}$ . Hence  $\ker(\psi)$  is the subgroup of  $\mathbb{Z}E^0$  generated by  $\{1_C : C \in \mathcal{C}\}$ . In particular, there is an isomorphism  $\theta_B : \ker(\psi) \rightarrow \mathbb{Z}\mathcal{C}$  satisfying  $1_C \mapsto \delta_C$ . For  $C \in \mathcal{C}$ , we have

$$(6.4) \qquad \qquad \qquad \phi(1_C) = \phi\left(\sum_{v \in C} \delta_v\right) = \sum_{v \in C} \delta_v - \sum_{r(e)=v} \delta_{s(e)}.$$

As  $v$  ranges over  $C$ , the vertices  $s(e)$  range over all vertices in orbits  $C' \in \mathcal{C}$  such that  $r^{-1}(C) \cap s^{-1}(C') \neq \emptyset$ . Moreover, for a fixed vertex  $v'$  on in an orbit  $C'$  with  $r^{-1}(C) \cap s^{-1}(C') \neq \emptyset$ , the term  $-\delta_{v'}$  occurs in the right-hand side of (6.4) precisely once for each edge  $e$  with  $r(e) \in C$  and  $s(e) = v'$ . By definition of the matrix  $B$ , these calculations establish that  $(1-B^t) \circ \theta_B = \theta_B \circ \phi|_{\ker(\psi)}$ . Hence  $\text{coker}(1-B^t) \cong \text{coker}(\phi)$  and  $\ker(1-B^t) \cong \ker(\phi)$  as required.  $\square$

*Remark 6.5.* Under the hypotheses of Proposition 6.4, the action  $\alpha$  of  $\mathbb{Z}$  on  $E$  is never free. Consequently, we cannot calculate the  $K$ -theory of  $C^*(E) \rtimes_{\alpha} \mathbb{Z}$  using the  $K$ -theory calculations for graph  $C^*$ -algebras and the quotient-graph construction of [9].

*Example 6.6.* As in Section 3 of [15], let  $\Lambda$  be a row-finite 2-graph with no sources such that  $\Lambda^{(\mathbb{N},0)}$  contains no cycles and each vertex  $v \in \Lambda^{0_2}$  is the range of an isolated cycle  $\Lambda^{(0,\mathbb{N})}$ .

By Proposition 3.6,  $\Lambda$  is isomorphic to  $\Lambda^{(\mathbb{N},0)} \rtimes_{\alpha} \mathbb{Z}$  where  $\alpha$  is determined by factorisations through paths in  $\Lambda^{(0,\mathbb{N})}$ . Since  $\Lambda^{(\mathbb{N},0)}$  has no cycles,  $C^*(\Lambda^{(\mathbb{N},0)})$  is an AF algebra and hence has trivial  $K_1$ -group. We can therefore apply Proposition 6.4 to obtain expressions for  $K_*(C^*(\Lambda))$ . In particular, our results generalise [15, Theorem 4.3(2)] to cover all 2-graphs described in Section 3 of [15].

## REFERENCES

- [1] R.J. Archbold and J.S. Spielberg, *Topologically free actions and ideals in discrete  $C^*$ -dynamical systems*, Proc. Edinb. Math. Soc. **37** (1994), 119–124.
- [2] B. Blackadar, *Shape theory for  $C^*$ -algebras*, Math. Scand. **56** (1985), 249–275.
- [3] N. Brownlowe, *Crossed products, endomorphisms and transfer operators*, Ph.D. Thesis, University of Newcastle, Australia, 2006.

- [4] D. G. Evans, *On the  $K$ -theory of higher-rank graph  $C^*$ -algebras*, preprint (2004) [arXiv:math.OA/0406458].
- [5] C. Farthing, *Removing sources from higher-rank graphs*, J. Operator Theory, to appear [arXiv:math.OA/0603037].
- [6] N. J. Fowler and A. Sims, *Product systems over right-angled Artin semigroups*, Trans. Amer. Math. Soc. **354** (2002), 1487–1509.
- [7] J.H. Hong and W. Szymański, *Quantum spheres and projective spaces as graph  $C^*$ -algebras*, Commun. Math. Phys. **232** (2002), 157–188.
- [8] J.A. Jeong, *Real rank of  $C^*$ -algebras associated with graphs*, J. Aust. Math. Soc. **77** (2004), 141–147.
- [9] A. Kumjian and D. Pask,  *$C^*$ -algebras of directed graphs and group actions*, Ergod. Th. & Dynam. Sys. **19** (1999), 1503–1519.
- [10] A. Kumjian and D. Pask, *Higher rank graph  $C^*$ -algebras*, New York J. Math. **6** (2000), 1–20.
- [11] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.
- [12] A. Kumjian, D. Pask and A. Sims,  *$C^*$ -algebras associated to coverings of  $k$ -graphs*, preprint 2006 [arXiv:math.OA/0612204].
- [13] D. Pask, J.C. Quigg, and I. Raeburn, *Coverings of  $k$ -graphs*, J. Algebra **289** (2005), 161–191.
- [14] D. Pask and I. Raeburn, *On the  $K$ -theory of Cuntz-Krieger algebras*, Publ. RIMS, Kyoto Univ. **32** (1996), 415–443.
- [15] D. Pask, I. Raeburn, M. Rørdam and A. Sims, *Rank-2 graphs whose  $C^*$ -algebras are direct limits of circle algebras*, **239** (2006), 137–178.
- [16] M. V. Pimsner and D. Voiculescu, *Exact sequences for  $K$ -groups and EXT-groups of certain cross-product  $C^*$ -algebras*, J. Operator Theory **4** (1980), 93–118.
- [17] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, **103**, Amer. Math. Soc., 2005.
- [18] I. Raeburn, A. Sims, and T. Yeend, *Higher-rank graphs and their  $C^*$ -algebras*, Proc. Edinb. Math. Soc. **46** (2003), 99–115.
- [19] I. Raeburn, A. Sims, and T. Yeend, *The  $C^*$ -algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. **213** (2004), 206–240.
- [20] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), 39–59.
- [21] D. Robertson and A. Sims, *Simplicity of higher-rank graph algebras*, Bull. London Math. Soc., to appear [arXiv:math.OA/0602120].
- [22] W. Szymański, *Simplicity of Cuntz-Krieger algebras of infinite matrices*, Pacific J. Math. **199** (2001), 249–256.
- [23] D. P. Williams, *Crossed products of  $C^*$ -algebras*, Math. Surveys and Monographs, vol. 134, Amer. Math. Soc., Providence, 1998.

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